Week 13: characteristic functions, central limit theorem

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 18/12/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1.

(1) Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables such that

$$\sqrt{n}(X_n - a) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\sigma^2)$$

with $a \in \mathbb{R}$ and $\sigma > 0$.

(a) Show that $X_n \to a$ in probability.

(b) Let $u : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to a} u(x) = 0$. Show that $\sqrt{n}(X_n - a)u(X_n) \to 0$ in probability.

Hint. First show that $u(X_n) \to 0$ in probability using the subsequence Lemma.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that it is differentiable at a with $g'(a) \neq 0$. Show that

$$\sqrt{n}(g(X_n) - g(a)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, g'(a)^2 \cdot \sigma^2).$$

Hint. Using Taylor's expansion, write $g(x) = g(a) + (x-a)g'(a) + (x-a)u(x)$ with u a function having limit 0 at a.

(2) Fix $p \in (0,1]$ and for $n \geq 1$ let X_n be a $\mathsf{Bin}(n,p)$ random variable. Show that

$$\sqrt{n}\left(\ln\left(\frac{X_n}{n}\right) - \ln(p)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, \frac{1-p}{p}\right).$$

Solution:

(1) (a) Denote by N a $\mathcal{N}(0,\sigma^2)$ random variable. Fix $\varepsilon > 0$. We have $\mathbb{P}(|X_n - a| \geq \varepsilon) = \mathbb{P}\left(\sqrt{n}|X_n - a| \geq \varepsilon\sqrt{n}\right)$. We can find $M > 0$ such that $\mathbb{P}(|N| \geq M) \leq \varepsilon$. Since the cdf of N is continuous, we have

$$\mathbb{P}\left(\sqrt{n}|X_n - a| \geq M\right) \underset{n\to\infty}{\longrightarrow} \mathbb{P}(|N| \geq M) \leq \varepsilon.$$

Thus for n sufficiently large

$$\mathbb{P}\left(\sqrt{n}|X_n - a| \geq M\right) \leq 2\varepsilon.$$

Also, for n sufficiently large $\varepsilon\sqrt{n} \geq M$. Thus for n sufficiently large:

$$\mathbb{P}\left(\sqrt{n}|X_n - a| \geq \varepsilon\sqrt{n}\right) \leq \mathbb{P}\left(\sqrt{n}|X_n - a| \geq M\right) \leq 2\varepsilon,$$

which gives the result.

(b) We show that $u(X_n) \to 0$ in probability by showing that for every subsequence ϕ there exists a subsequence ψ such that $u(X_{\phi(\psi(n))}) \to 0$ almost surely. To this end, observe that since $X_{\phi(n)}$ converges in probability to a, there exists a subsequence ψ such that $X_{\phi(\psi(n))}$ converges almost surely to a. Since $\lim_{x\to a} u(x) = 0$, this implies that $u(X_{\phi(\psi(n))}) \to 0$ almost surely.

Now, by Slutsky's theorem, $(\sqrt{n}(X_n - a), u(X_n)) \to (N, 0)$ in distribution, so $\sqrt{n}(X_n - a) \cdot u(X_n)$ converges in distribution to $N \cdot 0 = 0$, which is a constant and thus the convergence also holds in probability.

(c) Let u be as defined in the hint. Then write

$$\sqrt{n}(g(X_n) - g(a)) = \sqrt{n}(X_n - a)g'(a) + \sqrt{n}(X_n - a)u(X_n).$$

By assumption, $\sqrt{n}(X_n - a)g'(a) \to g'(a)\mathcal{N}(0,\sigma^2)$ in distribution, and $g'(a)\mathcal{N}(0,\sigma^2)$ has the same distribution as $\mathcal{N}(0, g'(a)^2 \cdot \sigma^2)$. By (b), $\sqrt{n}(X_n - a)u(X_n) \to 0$ in probability, and the desired result will follow from Slutsky's theorem.

(2) Let $(B_i)_{i\geq 1}$ be a sequence of i.i.d. random variables following a Bernoulli distribution of parameter p. We know that X_n and $B_1 + \cdots + B_n$ have the same law. Since the variance of B_1 is $p(1-p)$, by the central limit theorem we have

$$\frac{X_n - pn}{\sqrt{n}n} \overset{(d)}{\underset{n\to\infty}{\longrightarrow}} \mathcal{N}(0, p(1-p)).$$

Equivalently,

$$\sqrt{n}\left(\frac{X_n}{n} - p\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, p(1-p))$$

and the desired result follows from taking $g(x) = \ln(x)$ with $g'(p) = \frac{1}{p}$.

□

2 Training exercises

Exercise 2.

(1) Compute the characteristic function of an exponential random variable of parameter λ.

(2) Let $(X_i)_{1\leq i\leq n}$ be independent random variables such that X_i follows a Poisson(λ_i) distribution for every $1 \leq i \leq n$. Show that $X_1 + \cdots + X_n$ follows a Poisson($\lambda_1 + \lambda_2 + \cdots + \lambda_n$) random variable.

(3) Find the limit of $e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}$ as $n \to \infty$.

Hint. Use the central limit theorem.

Solution:

(1) Let X be an exponential random variable of parameter $\lambda > 0$. Using the transfer theorem, we get for $t \in \mathbb{R}$:

$$\mathbb{E}\left[e^{itX}\right] = \int_0^\infty e^{itx} \lambda e^{-\lambda x} dx = \int_0^\infty \lambda e^{-(\lambda - it)x} \mathrm{d}x = \frac{\lambda}{\lambda - it}.$$

(2) We argue using characteristic functions. We have seen in class that the characteristic function $\phi(t)$ of a Poi(λ) random variable is

$$\phi(t) = \exp(\lambda(e^{it} - 1)).$$

As a consequence, for $t \in \mathbb{R}$:

$$\phi_{X_1+\cdots+X_n}(t) = \mathbb{E}\left[e^{itX_1+\cdots+itX_n}\right] = \mathbb{E}\left[\prod_{k=1}^{n} e^{itX_k}\right] = \prod_{k=1}^{n} \mathbb{E}\left[e^{itX_k}\right] = \prod_{k=1}^{n} \exp(\lambda_k(e^{it} - 1)).$$

where we have used independence for the third equality. This quantity is equal to $\exp((\lambda_1 + \cdots + \lambda_n)(e^{it} - 1))$, and we recognize the characteristic function of a Poisson($\lambda_1 + \lambda_2 + \cdots + \lambda_n$) random variable.

(3) Observe that $e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}=\mathbb{P}(\mathsf{Poi}(n)\leq n)$. Thus by (2),

$$e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}=\mathbb{P}(P_1+\ldots+P_n\leq n),$$

where $P_1,\ldots,P_n$ are independent Poisson random variables of parameter 1 independent. And

$$\mathbb{P}(P_1+\ldots+P_n\leq n)=\mathbb{P}\left(\frac{P_1+\ldots+P_n-n}{\sqrt{n}}\leq 0\right).$$

Since the variance of a Poisson random variable with parameter λ is equal to λ, by the central limit theorem we have

$$\frac{P_1+\ldots+P_n-n}{\sqrt{n}}\xrightarrow[n\to\infty]{(\mathrm{d})}N,$$

where N is a standard $\mathcal{N}(0,1)$ Gaussian random variable. Now the cdf of N is continuous, so

$$\lim_{n\to\infty}e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}=\mathbb{P}(N\leq 0)=\frac{1}{2}.$$

□

Exercise 3. Let $(X_k)_{k\geq1}$ be a sequence of i.i.d. standand $\mathcal{N}(0,1)$ random variables. Set

$$Y_n=\frac{1}{n}\sum_{k=1}^{n}\sqrt{k}X_k.$$

Study the convergence in distribution of Y_n.

Solution:

We use Lévy's theorem. For $t\in\mathbb{R}$, we have

$$\mathbb{E}\left[e^{itY_n}\right]=\mathbb{E}\left[e^{\sum_{k=1}^{n}\frac{it\sqrt{k}}{n}X_k}\right]=\prod_{k=1}^{n}e^{-\frac{t^2}{2}\frac{k}{n^2}}=e^{-\frac{t^2}{2}\sum_{k=1}^{n}\frac{k}{n^2}}\quad\xrightarrow[n\to\infty]{}\quad e^{-\frac{t^2}{4}}.$$

We know that $e^{-\frac{t^2}{4}}$ is the characteristic function of a $\mathcal{N}(0,\frac{1}{2})$ random variable. Thus by Lévy's theorem we conclude that Y_n converges in distribution to $\mathcal{N}(0,\frac{1}{2})$. □

Exercise 4. Let $(X_n)_{n\geq1}$ be a sequence of i.i.d. centered random variables with $\mathbb{E}[X_1^2]\in(0,\infty)$. Show that the sequence given by

$$Y_n=\frac{\sum_{k=1}^{n}X_k}{1+\left(\sum_{k=1}^{n}X_k^2\right)^{1/2}}$$

converges in distribution as $n\to\infty$ and identify its limit.

Solution:

Denote by σ^2 the variance of X_1. Observe that

$$\frac{\sum_{k=1}^n X_k}{1+\left(\sum_{k=1}^n X_k^2\right)^{1/2}} = \frac{\sum_{k=1}^n X_k}{\sigma\sqrt{n}} \cdot \frac{\sigma\sqrt{n}}{1+\left(\sum_{k=1}^n X_k^2\right)^{1/2}} = \frac{\sum_{k=1}^n X_k}{\sigma\sqrt{n}} \cdot \frac{1}{\frac{1}{\sigma\sqrt{n}}+\left(\frac{\sum_{k=1}^n X_k^2}{\sigma^2 n}\right)^{1/2}}.$$

Setting

$$U_n = \frac{\sum_{k=1}^n X_k}{\sigma\sqrt{n}}, \qquad V_n = \frac{1}{\frac{1}{\sigma\sqrt{n}}+\left(\frac{\sum_{k=1}^n X_k^2}{\sigma^2 n}\right)^{1/2}},$$

the central limit and the strong law of large numbers respectively yield

$$U_n \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1), \qquad V_n \xrightarrow[n\to\infty]{a.s.} 1.$$

Thus, by Slutsky's theorem, $(U_n, V_n) \to (\mathcal{N}(0,1), 1)$ in distribution, and by continuous mapping this implies $U_n V_n \to \mathcal{N}(0,1)$ in distribution. □

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. real-valued random varirables. Assume that $\mathbb{E}[X_1^2] < \infty$. Set $m = \mathbb{E}[X_1]$, $\sigma^2 = \mathrm{Var}(X_1)$ and $Z_n = \frac{1}{\sqrt{n}}\sum_{k=1}^n (X_k - m)$.

(1) Recall the convergence in distribution of the sequence $(Z_n)_{n\geq 1}$.

(2) Show that $(Z_{2n} - Z_n)_{n\geq 1}$ converges in distribution and identity the limiting law.

Hint. Write $Z_{2n} - Z_n = aZ_n + bZ'_n$ for $a, b \in \mathbb{R}$ chosen in such a way that Z_n and Z'_n are independent and have the same law.

(3) Deduce that if $\sigma^2 > 0$, then the sequence $(Z_n)_{n\geq 1}$ does not converge in probability.

Remark. This shows that the convergence of the central limit theorem does not hold in probability.

Solution:

(1) By the central limit theorem, Z_n converges in distribution to σN, where N is a standard Gaussian $\mathcal{N}(0,1)$ random variable.

(2) We have

$$Z_{2n} - Z_n = \left(\frac{1}{\sqrt{2}} - 1\right) Z_n + \frac{1}{\sqrt{2}} Z'_n \qquad \text{avec} \qquad Z'_n = \frac{1}{\sqrt{n}} \sum_{k=n+1}^{2n} (X_k - m).$$

Since Z'_n and Z_n are independent and have the same law, we deduce that

$$\phi_{Z_{2n}-Z_n}(t) = \phi_{Z_n}\left(\left(\frac{1}{\sqrt{2}}-1\right)u\right)\cdot \phi_{Z_n}\left(\frac{1}{\sqrt{2}}u\right) \quad\longrightarrow\quad \phi_{\sigma N}\left(\left(\frac{1}{\sqrt{2}}-1\right)u\right)\cdot \phi_{\sigma N}\left(\frac{1}{\sqrt{2}}u\right)$$
$$= \exp\left(-\frac{u^2}{2}\sigma^2\left(\left(\frac{1}{\sqrt{2}}-1\right)^2+\frac{1}{2}\right)\right).$$

Thus $Z_{2n}-Z_n$ converges in distribution to $\sigma\sqrt{2}\sqrt{1-\frac{1}{\sqrt{2}}}N$.

(3) Assume that $\sigma^2>0$ and argue by contradiction by assuming that Z_n converges in probability to 0. Then the sequence $(Z_{2n}-Z_n)$ converges in probability to 0 (because then $(Z_n,Z_{2n})\to(0,0)$ in probability, and then we apply the continuous mapping $f(x,y)=x-y$). We conclude using the previous question that $\sigma=0$, which is a contradiction.

□

Exercise 6. (Riemann-Lebesgue Lemma) Let X be a real-valued random variable having density p.

(1) Show that for every $\varepsilon>0$ there exists a simple function g of the form $\sum_i c_i 1_{A_i}$, where the A_i are open intervals of $\mathbb{R}$, such that $\int_{\mathbb{R}}|p(x)-g(x)|dx<\varepsilon$.

(2) Show that the characteristic function φ of X satisfies

$$\lim_{t\to\pm\infty}\varphi(t)=0.$$

Solution:

(1) Since $\int_{\mathbb{R}\setminus[-M,M]}p(x)\mathrm{d}x\to 0$ as $M\to\infty$, without loss of generality we may work on $[-M,M]$. There exists a sequence f_n of simple functions such that $0\le f_n\uparrow p$ on $[-M,M]$. The convergence is pointwise and thus in L^1 by dominated convergence. Thus it enough to show that for $A\subset[-M,M]$ there exists a finite collection $(I_i)_{1\le i\le k}$ of open intervals such that

$$\int_{\mathbb{R}}|\mathbb{1}_A(x)-\mathbb{1}_{I_1\cup I_2\cup\cdots\cup I_k}(x)|\mathrm{d}x<\varepsilon. \tag{1}$$

To see this, by a general fact from measure theory, denoting by λ the Lebesgue measure, we can find an open set O such that $A\subset O$ and $\lambda(O\backslash A)\le\varepsilon/2$ (outer regularity of λ). Since O is open, we can write it is an at most countable union of pairwise disjoint intervals $O=\cup_{i\in I}I_i$. We can than find a finite subcollection such that $\lambda(O\backslash\cup_{1\le i\le k}I_i)\le\varepsilon/2$ and (1) follows.

(2) Let $\varepsilon>0$. Let $g=\sum_i c_i 1_{A_i}$, where the A_i are open intervals of $\mathbb{R}$, be such that $\int_{\mathbb{R}}|p(x)-g(x)|dx<\varepsilon$. Then, for every $t\in\mathbb{R}$

$$\left|\varphi(t)-\int_{\mathbb{R}}g(x)e^{itx}dx\right|<\varepsilon.$$

We observe that if $A_i = (a,b)$, then

$$\int_a^b e^{itx} dx = 2e^{it\frac{a+b}{2}} \cdot \frac{\sin(\frac{a-b}{2}t)}{t},$$

which goes to 0 as $|t| \to \infty$. Then, we can find M large enough such that for all t such that $|t| > M$,

$$\left| \int_{\mathbb{R}} g(x) e^{itx} dx \right| = \left| \sum_i c_i \int_{A_i} e^{itx} dx \right| < \varepsilon.$$

Therefore, for $|t| > M$, $|\varphi(t)| < \varepsilon$, which concludes the proof.

□

Exercise 7. Fix $\lambda > 1$ and let $(X_t)_{t \geq 0}$ be a family of random variables such that for every $t \geq 0$, X_t follows a geometric distribution with parameter $1 - e^{-t}$, that is

$$\mathbb{P}(X_t = k) = e^{-t}(1 - e^{-t})^{k-1}, \qquad k \geq 1.$$

Let $(U_n)_{n \geq 1}$ be a sequence of random variables such that $\lambda U_n - \ln(n)$ converges in probability to $-\ln(\mathcal{E})$ as $n \to \infty$, where $\mathcal{E}$ is an exponential random variable of parameter 1. Also assume that the two families $(X_t)_{t \geq 0}$ and $(U_n)_{n \geq 1}$ are independent.

Show that as $n \to \infty$, $X_{U_n}/n^{1/\lambda}$ converges in distribution to an exponential random variable, whose parameter is random and is equal to $\mathcal{E}^{1/\lambda}$.

Solution:

We use Lévy's theorem. To this end, we first compute the characteristic function of X_t:

$$\mathbb{E}\left[e^{iuX_t}\right] = \frac{1}{1 - e^t(1 - e^{-iu})}, \qquad u \in \mathbb{R}.$$

By independence of $(X_t)_{t \geq 0}$ and U_n, we thus have

$$\mathbb{E}\left[e^{iuX_{U_n}/n^{1/\lambda}}\right] = \mathbb{E}\left[\frac{1}{1 - e^{U_n}(1 - e^{-iu/n^{1/\lambda}})}\right].$$

By using a Taylor expansion we get

$$\frac{1}{1 - e^{U_n}(1 - e^{-iu/n^{1/\lambda}})} \quad \underset{n \to \infty}{\longrightarrow} \quad \frac{\mathcal{E}^{1/\lambda}}{\mathcal{E}^{1/\lambda} - iu}.$$

But

$$\forall s \geq 0, \quad \forall t \in \mathbb{R}, \qquad \left| \frac{1}{1 - e^s(1 - e^{-it})} \right| \leq 1.$$

By dominated convergence we get

$$\mathbb{E}\left[e^{iuX_{U_n}/n^{1/\lambda}}\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\frac{\mathcal{E}^{1/\lambda}}{\mathcal{E}^{1/\lambda}-iu}\right].$$

The result follows, because $x/(x-iu) = \mathbb{E}\left[e^{iu\mathsf{Exp}(x)}\right]$, where $\mathsf{Exp}(x)$ is an exponential random variable of parameter x. □

4 Fun exercise (optional, will not be covered in the exercise class)

You have a box with n red balls and n blue balls. You take out each time a ball at random but, if the ball was red, you put it back in the box and take out a blue ball. If the ball was blue, you put it back in the box and take out a red ball.

You keep doing it until left only with balls of the same color. What is the behavior of the number of balls that will be left as $n \to \infty$?

Solution:
If R_n denotes the number of balls left, it turns out that

$$\frac{R_n}{n^{3/4}} \overset{(d)}{\underset{n\to\infty}{\longrightarrow}} \left(\frac{8}{3}\right)^{1/4} \sqrt{|Z|},$$

where Z is a standard $\mathcal{N}(0,1)$ random variable, see https://pi.math.cornell.edu/~levine/erosion.pdf (it not at all easy to prove!). □