Week 1: σ-fields and measures

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 25/09/2023 17:00 (online) following the instructions on the course website:

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. **($\liminf$ and $\limsup$ of sets)**

Let (A_n) be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall the definitions $\liminf_{n\to\infty} A_n = \cup_{n\geq 1} \cap_{m\geq n} A_m$ and $\limsup_{n\to\infty} A_n = \cap_{n\geq 1} \cup_{m\geq n} A_m$.

(1) If $A = \mathbb{R}$, give the set $\limsup_{n\to\infty} A_n$ in the following three cases (please justify your answers):

(a) $A_n = [-1/n, 3+1/n]$

(b) $A_n = [-2-(-1)^n, 2+(-1)^{n+1})$

(c) $A_n = p_n\mathbb{N}$, where $(p_n)_{n\geq 1}$ is the sequence of prime numbers and $p_n\mathbb{N}$ denotes the set of all multiples of p_n.

(2) Show that

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathbb{P}(A_n) \leq \limsup_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}\left(\limsup_{n\to\infty} A_n\right).$$

Solution:

(1) We saw in the lecture that the set $\limsup_{n\to\infty} A_n$ represents the real numbers that appear in infinitely many sets A_n:

(a) We have $A_n = [0,3]$.

First solution. Observe that for every $n \geq 1$:

$$\bigcup_{m\geq n} A_m = [-1/n, 3+1/n].$$

We argue by double inclusion.

First, for every $m \geq n$ we have $[0,3] \subset A_m$, so $[0,3] \subset \cup_{m\geq n} A_m$, so $[0,3] \subset \cap_{n\geq 1} \cup_{m\geq n} A_m$.

Second, if $x \in \cap_{n\geq 1} \cup_{m\geq n} A_m$, then for every $n \geq 1$ we have $-1/n \leq x \leq 3+1/n$ and by passing to the limit as $n \to \infty$ we get $0 \leq x \leq 3$. Thus $\cap_{n\geq 1} \cup_{m\geq n} A_m \subset [0,3]$.

Second solution. We argue by double inclusion. Clearly if $0 \leq x \leq 3$ then x belongs to infinitely many A_n's (it actually belongs to all of them), so $[0,3] \subset \limsup_{n\to\infty} A_n$. Also, if $x \notin [0,3]$ that for n sufficiently large $x \notin A_n$, so x does not belong to infinitely many A_n's. Thus $x \notin \limsup_{n\to\infty} A_n$. This shows that $\limsup_{n\to\infty} A_n \subset [0,3]$.

For the next questions, we will use the more "intuitive" approach of the second fact, based on the fact that the set $\limsup_{n\to\infty} A_n$ represents the real numbers that appear in infinitely many sets A_n.

(b) We have $A_n = [-3,3)$. We show this equality between sets by double inclusion. Clearly, if $-3 \leq x < 3$, then x belongs to infinitely many A_n's (if $-1 \leq x < 3$, $x \in A_{2n+1}$ for every $n \geq 1$ and if $-3 \leq x \leq 1$ then $x \in A_{2n}$ for every $n \geq 1$), so $[-3,3) \subset \limsup_{n\to\infty} A_n$. If $x \notin [-3,3)$ than there is no n such that $x \in A_n$, so $x \notin \limsup_{n\to\infty} A_n$. This shows that $\limsup_{n\to\infty} A_n \subset [-3,3)$.

(c) We have $A_n = \{0\}$. We show this equality between sets by double inclusion. Cleary $0 \in p_n\mathbb{N}$ for every $n \geq 1$, so $\{0\} \subset p_n\mathbb{N}$. Conversively, if $m \in \limsup_{n\to\infty} A_n$, then m is divisible by infinitely many prime numbers, which implies $m = 0$. Thus $\limsup_{n\to\infty} A_n \subset \{0\}$.

(2) Recall from the lecture the following fact:

$$\text{if } B_n \subset B_{n+1} \text{ for all } n \geq 1, \text{then} \quad \mathbb{P}(B_n) \underset{n\to\infty}{\longrightarrow} \mathbb{P}(\cup_m B_m). \tag{1}$$

By (1) applied to the sequence $B_n = \cap_{m\geq n} A_m$ we get

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(\cap_{m\geq n} A_m).$$

For every $m' \geq n$ we have $\cap_{m\geq n} \subset A_{m'}$, so $\mathbb{P}(\cap_{m\geq n} A_m) \leq \mathbb{P}(A_{m'})$ and hence $\mathbb{P}(\cap_{m\geq n} A_m) \leq \inf_{m\geq n} \mathbb{P}(A_m)$. Therefore,

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(\cap_{m\geq n} A_m) \leq \lim_{n\to\infty} \inf_{m\geq n} \mathbb{P}(A_m)$$

and the right hand side is exactly the definition of the inferior limit.

The middle inequality of the statement is trivial.

On could show the final inequality as the first one, but we use the very useful complementation trick by applying the first inequality to the sequence (A_n^c):

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n^c\right) \leq \liminf_{n\to\infty} \mathbb{P}(A_n^c).$$

Then using the fact that $\liminf_{n\to\infty} A_n^c = (\limsup_{n\to\infty} A_n)^c$ and $\liminf_{n\to\infty}(-x_n) = -\limsup_{n\to\infty}(x_n)$, we thus have

$$1-\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) \le \mathbb{P}\left(\liminf_{n\to\infty} A_n^c\right) \le \liminf_{n\to\infty} \mathbb{P}(A_n^c) = \liminf_{n\to\infty}(1-\mathbb{P}(A_n)) = 1-\limsup_{n\to\infty} \mathbb{P}(A_n),$$

and the result follows.

□

2 Training exercises

Exercise 2. **(0 or 1)**
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

(1) Let A and B two events with $A \subset B$. Assume that $\mathbb{P}(A) = 1$; what can be said of $\mathbb{P}(B)$? And if $\mathbb{P}(B) = 0$, what can be said of $\mathbb{P}(A)$?

Let $(A_i)_{i\ge1}$ be a sequence of events.

(2) Assume that $\mathbb{P}(A_i) = 0$ for every $i \ge 1$. Show that $\mathbb{P}\left(\cap_{i=1}^\infty A_i\right) = 0$ and that $\mathbb{P}\left(\cup_{i=1}^\infty A_i\right) = 0$.

(3) Assume that $\mathbb{P}(A_i) = 1$ for every $i \ge 1$. Show that $\mathbb{P}\left(\cup_{i=1}^\infty A_i\right) = 1$ and that $\mathbb{P}\left(\cap_{i=1}^\infty A_i\right) = 1$.

Solution:
We recall that if $A \subset B$ are two events, then $\mathbb{P}(A) \le \mathbb{P}(B)$, and if $(A_i)_{i\ge1}$ is a sequence of events, we have $\mathbb{P}(\cup_{i\ge1} A_i) \le \sum_{i\ge1} \mathbb{P}(A_i)$.

(1) If $\mathbb{P}(A) = 1$, then $\mathbb{P}(B) \ge 1$, and since $\mathbb{P}(B) \in [0,1]$, we have $\mathbb{P}(B = 1)$. If $\mathbb{P}(B) = 0$, then $\mathbb{P}(A) \le 0$, and since $\mathbb{P}(A) \in [0,1]$, we have $\mathbb{P}(A) = 0$.

(2) We have $\cap_{i=1}^\infty A_i \subset A_1$ donc $\mathbb{P}\left(\cap_{i=1}^\infty A_i\right) \le \mathbb{P}(A_1) = 0$. Thus $\mathbb{P}\left(\cap_{i=1}^\infty A_i\right) = 0$. Similarly

$$\mathbb{P}\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_{i=1}^\infty \mathbb{P}(A_i) = \sum_{i=1}^\infty 0 = 0.$$

Thus $\mathbb{P}\left(\cup_{i=1}^\infty A_i\right) = 0$.

(3) We have $\mathbb{P}(A_i) = 0$ for every $i \ge 1$. We get the result by taking the complementary event and using question (1);

$$\mathbb{P}\left(\cup_{i=1}^\infty A_i\right) = 1-\mathbb{P}\left(\cap_{i=1}^\infty A_i^c\right) = 1, \qquad \mathbb{P}\left(\cap_{i=1}^\infty A_i\right) = 1-\mathbb{P}\left(\cup_{i=1}^\infty A_i^c\right) = 1$$

□

Exercise 3. **(Generating π-systems)** A collection of sets $\mathcal{A}$ is said to be a generating π-system of a σ-field $\mathcal{B}$ if $\sigma(\mathcal{A}) = \mathcal{B}$ and if $\mathcal{A}$ is closed under finite intersections (meaning that for every $A, B \in \mathcal{A}$ we have $A \cap B \in \mathcal{A}$).

(1) Show that $\mathcal{A} = \{[0,a] : a \in [0,1]\}$ is a π-system generating $\mathcal{B}([0,1])$.

(2) Prove that $\mathcal{A}' = \{(-\infty, a_1] \times \cdots \times (-\infty, a_d] \colon a_1, \ldots, a_d \in \mathbb{R}\} \cup \{\mathbb{R}^d\}$ is a π-system generating the σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

Solution:

(1) The system of sets is stable under intersections since $[0,a] \cap [0,b] = [0, a \wedge b] \in \mathcal{A}$ for $a, b \in [0,1]$. Hence $\mathcal{A}$ is a π-system on $[0,1]$.

To see that $\mathcal{A}$ generates $\mathcal{B}([0,1])$, note first that for each $a \in [0,1]$ we have $[0,a] \in \mathcal{B}([0,1])$ since $[0,a]$ is closed in $[0,1]$. Hence $\mathcal{A} \subset \mathcal{B}([0,1])$ and thus $\sigma(\mathcal{A}) \subset \mathcal{B}([0,1])$. Conversely, if $U \subset [0,1]$ is open, then it is a union of sets of the form $[0,a]$ or $[0,a] \setminus [0,b]$ with $a, b \in [0,1]$. This yields the reverse inclusion $\mathcal{B}([0,1]) \subset \sigma(\mathcal{A})$.

(2) The system is stable under intersections since for $a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{R}$ we have

$$\begin{aligned}&((-\infty, a_1] \times \cdots \times (-\infty, a_d]) \cap ((-\infty, b_1] \times \cdots \times (-\infty, b_d])\\ &\quad = (-\infty, a_1 \wedge b_1] \times \cdots \times (-\infty, a_d \wedge b_d] \in \mathcal{A}'.\end{aligned}$$

Hence $\mathcal{A}'$ is a π-system on $\mathbb{R}^d$.

To see that $\mathcal{A}'$ generates $\mathcal{B}(\mathbb{R}^d)$, note that

$$(-\infty, a_1] \times \cdots \times (-\infty, a_d] = \bigcap_{n \geq 1} (-\infty, a_1 + 1/n) \times \cdots \times (-\infty, a_d + 1/n)$$

and that all the sets in the intersection are open. This implies that $\mathcal{A}' \subset \mathcal{B}(\mathbb{R}^d)$. Conversely, fix $U \subset \mathbb{R}^d$. First of all, one uses the definition of $\mathcal{A}'$ to see that $(a_1, b_1] \times \cdots \times (a_d, b_d] \in \sigma(\mathcal{A}')$ whenever $a, b \in \mathbb{R}^d$. Hence

$$U = \bigcup_{\substack{n \geq 0,\, k \in 2^{-n}\mathbb{Z}^d: \\ k + (-2^{-n}, 0]^d \subset U}} (k + (-2^{-n}, 0]^d) \in \sigma(\mathcal{A}')$$

Since the open set U was arbitrary, we get $\mathcal{B}(\mathbb{R}^d) \subset \sigma(\mathcal{A}')$ as required.

□

Exercise 4. (Questions and operations on σ-fields)

(1) Is the set of all open sets of $\mathbb{R}$ a σ-field?

(2) For every $n \geq 0$, define on $\mathbb{N}$ the σ-field $\mathcal{F}_n = \sigma(\{0\}, \{1\}, \ldots, \{n\})$. Show that the sequence of σ-fields $(\mathcal{F}_n, n \geq 0)$ is non-decreasing but that $\bigcup_{n \geq 0} \mathcal{F}_n$ is not a σ-field.
Hint: argue by contradiction and use the subset of even integers.

(3) We throw two coins. To model the outcome, we use the probability space $\Omega = \{00, 01, 10, 11\}$ equiped with the σ-field $\mathcal{P}(\Omega)$. Let $\mathbb{P}$ be the probability measure on Ω corresponding to the case where

the two coins are fair and are thrown independently. Let $\mathbb{Q}$ be the probability measure on Ω corresponding to the case where the second coin is rigged and always gives the same result as the first one. Show that the set $\{A \in \mathcal{P}(\Omega) : \mathbb{P}(A) = \mathbb{Q}(A)\}$ is not a σ-field (this gives in particular an example of a Dynkin system which is not a σ-field).

(4) Let $(E \times F, \mathcal{A})$ be a measured space and $\pi : E \times F \longrightarrow E$ the canonical projection defined by $\pi(x, y) = x$. Is the set $\mathcal{A}_E := \{\pi(A), A \in \mathcal{A}\}$ always a σ-field?

(5) Let $(E, \mathcal{A})$ be a measurable space. Let $\mathcal{C}$ be a collection of subsets of E, and fix $B \in \sigma(\mathcal{C})$. Alexandra says: there always exists a countable collection $\mathcal{D} \subset \mathcal{C}$ such that $B \in \sigma(\mathcal{D})$. Is she correct?

Solution:

(1) No, because the complement of $]-\infty, 0[$ is not open.

(2) The fact that the sets are nondecreasing comes from the fact that if $\mathcal{A} \subset \mathcal{B}$, then $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$. Set

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n,$$

and assume that $\mathcal{F}$ is a σ-field. We have

$$\{2n\} \in \mathcal{F}_{2n} \subset \mathcal{F} \quad \text{and} \quad 2\mathbb{N} = \bigcup_{n \geq 0} \{2n\}.$$

Therefore, $2\mathbb{N} \in \mathcal{F}$ *i.e.* there exists $n_0 \in \mathbb{N}$ such that $2\mathbb{N} \in \mathcal{F}_{n_0}$. But the only sets of $\mathcal{F}_{n_0}$ which have infinitely many elements are of the form $\mathbb{N} \setminus A$, where A is a subset of $\{0, 1, \ldots, n_0\}$. Indeed, elements of $\mathcal{F}_n$ are of the form A or $A \cup \{n+1, n+2, \ldots\}$ with $A \subset \{0, 1, \ldots, n\}$ (to see it, one checks that such elements form a σ-field containing $\{1\}, \{2\}, \ldots, \{n\}$, and that conversely any σ-field containing $\{1\}, \{2\}, \ldots, \{n\}$ contains all elemets of this type).

This gives a contradiction.

(3) We equip Ω with the σ-field $\mathcal{P}(\Omega)$. The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are given by

$$\mathbb{P}(00) = \mathbb{P}(01) = \mathbb{P}(10) = \mathbb{P}(11) = \frac{1}{4}, \qquad \mathbb{Q}(00) = \mathbb{Q}(11) = \frac{1}{2}.$$

Then $\quad \{A \in \Omega : \mathbb{P}(A) = \mathbb{Q}(A)\}$ is equal to

$$\{\{00, 01\}, \{00, 10\}, \{11, 01\}, \{11, 10\}, \varnothing, \Omega\},$$

which is not a σ-field since it is not stable by intersections.

(4) Take $E = F = \{0,1\}$ and consider the σ-field $\mathcal{F}$ generated by the element $(0,0) \in E \times F$. It is clear that

$$\mathcal{F} = \{\varnothing, E \times F, \{(0,0)\}, E \times F \backslash \{(0,0)\}\}.$$

We check that $\mathcal{F}_E = \{\varnothing, \{0\}, E\}$, which is not a σ-field.

(5) Alexandra is correct. Indeed, set

$$\mathcal{G} = \{B \in \sigma(\mathcal{C}) ; \exists \mathcal{D} \subset \mathcal{C} \text{ countable such that } B \in \sigma(\mathcal{D})\}.$$

Let us show that $\mathcal{G}$ is a σ-field.

It is clear that $E \in \mathcal{G}$.

If $A \in \mathcal{G}$, then there exists $\mathcal{D} \subset \mathcal{C}$ coutable such that $A \in \sigma(\mathcal{D})$, so $A^c \in \sigma(\mathcal{D})$: we have $A^c \in \mathcal{G}$.

If $(A_n) \subset \mathcal{G}$, then for every n there exists $\mathcal{D}_n \subset \mathcal{C}$ countable such that $A_n \in \sigma(\mathcal{D}_n)$, so $\cup_n A_n \in \sigma(\mathcal{D})$, where $\mathcal{D} := \cup_n D_n \subset \mathcal{C}$ is countable (being a countable union of countable sets): we have $\cup_n A_n \in \mathcal{G}$.

We conclude that $\mathcal{G}$ is a σ-field.

But $\mathcal{C} \subset \mathcal{G}$, which implies that $\sigma(\mathcal{C}) \subset \sigma(\mathcal{G}) = \mathcal{G} \subset \sigma(\mathcal{C})$, hence the result.

Remark. One could be led to think that $\sigma(\mathcal{C})$ can be explicitely constructed from $\mathcal{C}$ by adding all the countable unions of elements of $\mathcal{C}$ and of their complements, and then by iterating infinitely many times. In general, the result obtained in such a way is strictly smaller than $\sigma(\mathcal{C})$.

□

3 More involved exercise (optional, will not be coverd in the exercise class)

Exercise 5. Let $(E, \mathcal{E}, \mu)$ be a measured space with μ finite. Let $\mathcal{A}$ be a collection of subsets such that:
(a) $E \in \mathcal{A}$ (b) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (c) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ (d) $\sigma(\mathcal{A}) = \mathcal{E}$.

The goal of this exercise is to show that for every $E \in \mathcal{E}$, for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(E \Delta A) \leq \epsilon$ (where $X \Delta Y = (X \cup Y) \backslash (X \cap Y)$). To this end, set $\mathcal{S} = \{E \in \mathcal{E} : \forall \epsilon > 0, \exists A \in \mathcal{A} : \mu(E \Delta A) \leq \epsilon\}$.

(1) Show that $\mathcal{S}$ is stable by finite unions.

(2) Show that $\mathcal{S}$ is a σ-field (for stability by countable unions, justify that one may assume that the events are pairwise disjoint).

(3) (*Application to percolation*)

(a) Set $E = \{0,1\}^{\mathbb{Z}^2}$. We see the elements of E as $\mathbb{Z}^2$, with each vertex either occupied (value 1), or inoccupied (value 0). Denote by $B_n = \{x \in \mathbb{Z}^2 : \|x\|_\infty \leq n\}$ the "box" of size n. We say that $A \subset E$ is a cylinder of size n if there exists $(u_s) \in \{0,1\}^{B_n}$ such that

$$A = \{(x_s) \in E : x_s = u_s \, \forall s \in B_n\}.$$

Denote by C_n the set of cylinders of size n. We then take

$$\mathcal{E} = \sigma\left(\bigcup_{n \geq 1} C_n \right).$$

Connect two vertices at distance 1 with an edge if they are both occupied. Justify that the event "there exists an infinite path between connected vertices" is measurable (that it, in $\mathcal{E}$).

(b) Every vertex of $\mathbb{Z}^2$ is occupied independently with probability $p \in [0,1]$. For $x \in \mathbb{Z}^2$, denote by τ_x the translation by vector x. If $A \subset \{0,1\}^{\mathbb{Z}^2}$ is an event, we write $\tau_x A = \{(u_s) \in \{0,1\}^{\mathbb{Z}^2} : \tau_x^{-1}(u_s) \in A\}$. Let A be an event invariant under translations (that is $\tau_x A = A$ for every $x \in \mathbb{Z}^2$). Show that $\mathbb{P}(A)$ is 0 or 1. In particular, the probability of having an infinite connected component (in the sense of question (a)) is 0 or 1.

Solution:

(1) Take $A, B \in \mathcal{S}$. Fix $\epsilon > 0$ and consider $A', B' \in \mathcal{A}$ such that $\mu(A \Delta A') \leq \epsilon$ and $\mu(B \Delta B') \leq \epsilon$. But $(A \cup B)\Delta(A' \cup B') \subset (A \Delta A') \cup (B \Delta B')$, so $\mu((A \cup B)\Delta(A' \cup B')) \leq 2\epsilon$.

(2) First of all, $E \in \mathcal{S}$ since $E \in \mathcal{A}$.

The class $\mathcal{S}$ is stable by complementation since $E \Delta A = E^c \Delta A^c$ and $\mathcal{A}$ is stable by complementation.

Let us show that $\mathcal{S}$ is stable by countable union. Thanks to question (1), we can consider a sequence (A_n) of disjoint elements of $\mathcal{S}$. Fixe $\epsilon > 0$ eand consider $A'_n \in \mathcal{A}$ such that $\mu(A_n \Delta A'_n) \leq \epsilon 2^{-n}$ for every $n \geq 1$. There exists N such that $\mu\left(\bigcup_{j \geq N+1} A_j\right) \leq \varepsilon/2$ (since the (A_i) are disjoint). Set $A' := \bigcup_{j=1}^N A'_j \in \mathcal{A}$. Since

$$\left(\bigcup_{k \geq 1} A_k \right) \Delta A' \subset \bigcup_{j=1}^{N} (A_j \Delta A'_j) \cup \bigcup_{k \geq N+1} A_k,$$

we are done.

(3) (a) Since a countable union of measurable sets is measurable, it suffices to check that A = "the origin is in an infinite connected component" is measurable. To this end, note that this event is equal to $B = \cap_{n \geq 1}$"0 is connected to the boundary of B_n". Indeed, A is clearly a subset of B. Conversely, if 0 is connected to the boundary of B_n, then the connected component of 0 has size at least n, and if this is true for every $n \geq 1$, then the size of the connected component of the origin is infinite. Hence $B \subset A$.

Remark. It is sometimes useful to have an explicit description of a family generated a σ-field, as will be seen in the next question. The advantage of working with cylinders is that they form a π-system (the interesection of two cylinders is always a cylinder).

(b) Let us consider the class $\mathcal{A}$ made of finite unions of cylinders. This class satisfies conditions (a), (b), (c), (d) (note that we cannot just take the class of cylinders, because the class of cylinders is not stable by finite unions). Intuitively, events of $\mathcal{A}$ are events which depend only on a finite number of sites.

Let A be an event invariant under translations. Fix $\epsilon > 0$. By what precedes, there exists an event B which depends only on a finite number of sites such that $\mathbb{P}(A\Delta B) \leq \epsilon$. It is then possible to find a vector x such that the translated event $\tau_x B$ uses different sites of B, so that the two events B and $\tau_x B$ are independent. Hence $\mathbb{P}(B \cap \tau_x B) = \mathbb{P}(B)^2$. But then

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A \cap \tau_x A) \leq \mathbb{P}(B \cap \tau_x B) + 2\epsilon = \mathbb{P}(B)^2 + 2\epsilon \leq \mathbb{P}(A)^2 + 4\epsilon.$$

Since this is true for every $\epsilon > 0$, we deduce that $\mathbb{P}(A) \leq \mathbb{P}(A)^2$, so $\mathbb{P}(A)$ is equal to 0 or to 1.

□

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 6. (The duelling idiots, taken from: *Duelling idiots and other probability puzzlers* P.J.Nahin, Princeton Univ.Press (2000)).

A and B decide to duel but they have just one gun (a six shot revolver) and only one bullet. Being dumb, this does not deter them and they agree to "duel" as follows: They will insert the lone bullet into the gun's cylinder, A will then spin the cylinder and shoot at B (who, standing inches away, is impossible to miss). If the gun doesn't fire then A will give the gun to B, who will spin the cylinder and then shoot at A. This back and forth duel will continue until one fool shoots the other.

What is the probability that A will win?

Solution:

We condition on the first try:

$$\begin{aligned}
\mathbb{P}(A \text{ wins}) &= \mathbb{P}(A \text{ wins} \cap \text{ 1st try of } A \text{ succeeds}) + \mathbb{P}(A \text{ wins} \cap \text{ 1st try of } A \text{ fails}) \\
&= \mathbb{P}(\text{ 1st try succeeds}) + \mathbb{P}(A \text{ wins} \cap \text{ 1st try of } A \text{ fails} \cap \text{ 1st try of } B \text{ fails}) \\
&= 1/6 + \mathbb{P}(A \text{ wins} \mid \text{1st try of } A \text{ fails} \cap \text{ 1st try of } B \text{ fails})) \times \mathbb{P}(\text{ 1st try of } A \text{ fails} \cap \text{ 1st try of } B \text{ fails}) \\
&= 1/6 + \mathbb{P}(A \text{ wins}) \times 5/6 \times 5/6,
\end{aligned}$$

since after two misses the whole process starts again. Finally we solve this last equation and find

$$\mathbb{P}(A \text{ wins}) = 6/11 = 0.545454\ldots$$

□