Week 2: Dynkin Lemma, independent σ-fields

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 2/10/2023 17:00 (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/`

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

* * *

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ be collections of measurable sets. Assume that $\mathcal{A}$ and $\mathcal{B}$ are stable by finite intersections and that for every $A \in \mathcal{A}, B \in \mathcal{B}$: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. Show that for every $U \in \sigma(\mathcal{A})$ and $V \in \sigma(\mathcal{B})$ we have: $\mathbb{P}(U \cap V) = \mathbb{P}(U) \cdot \mathbb{P}(V)$.

Hint: mimick the proof of the Dynkin Lemma by first introducing

$$G_1 = \{U \in \mathcal{F};\ \forall B \in \mathcal{B}, \mathbb{P}(U \cap B) = \mathbb{P}(U) \cdot \mathbb{P}(B)\}$$

and checking that it is a Dynkin system.

2 Training exercises

Exercise 2. **(Independences)** Alix has four books: a mathematics book, a biology book, a chemistry book and a mathematics-biology-chemistry book. Alix chooses one of the four books at random, with uniform probability. Denote by M, B and C the events "the chosen book has mathematics in it" (respectively biology, chemistry). Are the events M, B and C independent?

Exercise 3. **(Cylinders)** Sasha models coin tosses as follows. Let $\Omega = \{0,1\}^{\{1,2,3,\ldots\}}$, so that an element of Ω is a sequence of 0 and 1's. For $\omega = (\omega_n)_{n \geq 1} \in \Omega$ we interpret ω_k as the result of the k-th throw (1 for heads, 0 for tails). For all $k \geq 1$ and $u_1, \ldots, u_k \in \{0,1\}$ we define the following set, called a cylinder:

$$C_{u_1,u_2,\ldots,u_k} = \{(\omega_n)_{n \geq 1} : \omega_1 = u_1, \ldots, \omega_k = u_k\}, \tag{1}$$

(1) Express (using unions, intersections and complements) the following events in terms of sets of type (1) :
  (a) B_n: "We get tails for the first time on the nth throw"
  (b) A : "The result of the second throw is tails".
  (c) C: "You never get tails".
  (d) D_n: "you get tails at least twice in the first n throws".

We assume the existence the existence of a probability $\mathbb{P}$ on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is the σ-field generated by sets of the form (1) (cylinder σ-algebra) such that

$$\mathbb{P}\left(C_{u_1,u_2,\ldots,u_k}\right) = \frac{1}{2^k}. \tag{2}$$

(2) Compute the probabilities of the previous events A, B_n, C, D_n.

Exercise 4. Let $(A_n)_{n\geq 1}$ be a sequence of independent events on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that

$$\mathbb{P}\left(\bigcap_{n\geq 1} A_n\right) = \prod_{n\geq 1} \mathbb{P}(A_n).$$

Exercise 5. Let $(\mathcal{F}_n)$ be a sequence of independent σ-fields and consider a bijection $\sigma\colon \{1,2,3,\ldots\} \to \{1,2,3,\ldots\}$. Show that $(\mathcal{F}_{\sigma(n)})$ is still a sequence of independent σ-fields.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. Fix $\alpha > 0$, $a \in \{0,1\}^k$ and let $k_* = a_1 + \cdots + a_k$. Now consider a sequence of independent events (A_n) with $\mathbb{P}(A_n) = 1/n^\alpha$ for all $n \in \mathbb{N}$ and let

$$N = \#\{n \in \mathbb{N} \colon (\mathbb{1}_{A_n}, \mathbb{1}_{A_{n+1}}, \ldots, \mathbb{1}_{A_{n+k-1}}) = a\}.$$

If $\alpha k_* > 1$ show that $N < \infty$ almost surely. If $\alpha k_* \leq 1$ show that $N = \infty$ almost surely.

Exercise 7. **(Diophantine approximation and Borel-Cantelli)** We denote by λ the Lebesgue measure and work on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$.

(1) Let $\epsilon > 0$ be fixed. Show that

$$\lambda\left(\left\{x \in [0,1] : \exists \text{ an } \textbf{infinite} \text{ number of rationals } p/q \text{ with } \gcd(p,q) = 1 \text{ s.t. } \left|x - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}}\right\}\right) = 0.$$

Thus, almost all x are "badly approximated by rationals at order $2+\epsilon$".

Indication. For any $q \geq 1$, consider

$$A_q := [0,1] \cap \bigcup_{p=0}^{q} \left[\frac{p}{q} - \frac{1}{q^{2+\epsilon}}, \frac{p}{q} + \frac{1}{q^{2+\epsilon}}\right].$$

(2) Show that

$$\lambda\left(\left\{x \in [0,1] : \exists \text{ an } \textbf{infinite} \text{ number of rationals } p/q \text{ with } \gcd(p,q) = 1 \text{ s.t. } \left|x - \frac{p}{q}\right| \leq \frac{1}{q^{2}}\right\}\right) = 1.$$

Thus, almost all x are "well approximated by rationals at order 2".

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 8. The names of 100 mathematicians are placed in 100 wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the mathematicians are led into the room; each may look in at most 50 boxes, but must leave the room exactly as she found it and is permitted no further communication with the others. The mathematicians have a chance to plot their strategy in advance, and they are going to need it, because unless every single mathematician finds her own name all will subsequently lose their funding. Find a strategy for them which has probability of success (mathematics survive) exceeding 30%.

Remark. If each mathematician examines a random set of 50 boxes, their probability of success is $\frac{1}{2^{100}}$ (each mathematician that opens 50 boxes at random among 100 has a probability $\frac{1}{2}$ to find her name), which is very very small.