Week 2: Dynkin Lemma, independent σ-fields

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 2/10/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

***

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ be collections of measurable sets. Assume that $\mathcal{A}$ and $\mathcal{B}$ are stable by finite intersections and that for every $A \in \mathcal{A}, B \in \mathcal{B}$: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. Show that for every $U \in \sigma(\mathcal{A})$ and $V \in \sigma(\mathcal{B})$ we have: $\mathbb{P}(U \cap V) = \mathbb{P}(U) \cdot \mathbb{P}(V)$.

Hint: mimick the proof of the Dynkin Lemma by first introducing

$$G_1 = \{U \in \mathcal{F};\ \forall B \in \mathcal{B}, \mathbb{P}(U \cap B) = \mathbb{P}(U) \cdot \mathbb{P}(B)\}$$

Solution:

First step. Introduce

$$G_1 = \{U \in \mathcal{F};\ \forall B \in \mathcal{B}, \mathbb{P}(U \cap B) = \mathbb{P}(U) \cdot \mathbb{P}(B)\}.$$

We check that G_1 is a λ-system containing $\mathcal{A}$, which is stable by finite intersections. Indeed:

- $\Omega \in G_1$ since $\mathbb{P}(\Omega) = 1$.
- If $U \in G_1$, we check that $U^c \in G_1$. To this end, take $B \in \mathcal{B}$ and write

$$\mathbb{P}(U^c \cap B) = \mathbb{P}(B) - \mathbb{P}(U \cap B) = \mathbb{P}(B) - \mathbb{P}(U)\mathbb{P}(B) = \mathbb{P}(U^c)\mathbb{P}(B).$$

- If $(U_n)_{n\geq 1}$ is a sequence of pairwise disjoint events in G_1, we check that $\cup_n U_n \in G_1$. To this end, take $B \in \mathcal{B}$ and write

$$\mathbb{P}\left(\left(\bigcup_{n\geq 1} U_n\right) \cap B\right) = \mathbb{P}\left(\bigcup_{n\geq 1} (U_n \cap B)\right) = \sum_{n\geq 1} \mathbb{P}(U_n \cap B) = \sum_{n\geq 1} \mathbb{P}(U_n)\mathbb{P}(B) = \mathbb{P}\left(\bigcup_{n\geq 1} U_n\right)\mathbb{P}(B).$$

The second and last equalities comes from the fact that the measure of a union of pairwise disjoint events is the sum of their measures. The third equality comes from the fact that $U_n \in G_1$. Hence, by Dynkin's Lemma, $\sigma(\mathcal{A}) = \lambda(\mathcal{A}) \subset G_1$.

Second step. Introduce

$$G_2 = \{V \in \mathcal{F};\ \forall U \in \sigma(A), \mathbb{P}(U \cap V) = \mathbb{P}(U) \cdot \mathbb{P}(V)\}.$$

We similarly check that G_2 is a λ-system containing $\mathcal{B}$ (by the first step), which is stable by finite intersections. So, by by Dynkin's Lemma, $\sigma(\mathcal{B}) = \lambda(G_2) \subset G_2$, which completes the proof. □

2 Training exercises

Exercise 2. **(Independences)** Alix has four books: a mathematics book, a biology book, a chemistry book and a mathematics-biology-chemistry book. Alix chooses one of the four books at random, with uniform probability. Denote by M, B and C the events "the chosen book has mathematics in it" (respectively biology, chemistry). Are the events M, B and C independent?

Solution:
To model this problem, consider the probability space $\Omega = \{m, b, c, mbc\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ the uniform probability on Ω, so that $M = \{m, mbc\}$, $B = \{b, mbc\}$ and $C = \{c, mbc\}$. Note that $M \cap B = M \cap C = B \cap C = \{mbc\}$. So

$$\mathbb{P}(M \cap B) = \frac{1}{4} = \mathbb{P}(M)\mathbb{P}(B), \quad \mathbb{P}(M \cap C) = \frac{1}{4} = \mathbb{P}(M)\mathbb{P}(C), \quad \mathbb{P}(B \cap C) = \frac{1}{4} = \mathbb{P}(B)\mathbb{P}(C).$$

Thus, the events M, B, C are pairwise independent.

However,

$$\mathbb{P}(M \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = \mathbb{P}(M)\mathbb{P}(B)\mathbb{P}(C),$$

so events M, B, C are not (mutually) independent. □

Exercise 3. **(Cylinders)** Sasha models coin tosses as follows. Let $\Omega = \{0,1\}^{\{1,2,3,\dots\}}$, so that an element of Ω is a sequence of 0 and 1's. For $\omega = (\omega_n)_{n \geq 1} \in \Omega$ we interpret ω_k as the result of the k-th throw (1 for heads, 0 for tails). For all $k \geq 1$ and $u_1, \dots, u_k \in \{0,1\}$ we define the following set, called a cylinder:

$$C_{u_1, u_2, \dots, u_k} = \{(\omega_n)_{n \geq 1} : \omega_1 = u_1, \dots, \omega_k = u_k\}, \tag{1}$$

(1) Express (using unions, intersections and complements) the following events in terms of sets of type (1) :
  (a) B_n: "We get tails for the first time on the nth throw"
  (b) A : "The result of the second throw is tails".
  (c) C: "You never get tails".
  (d) D_n: "you get tails at least twice in the first n throws".

We assume the existence the existence of a probability $\mathbb{P}$ on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is the σ-field generated by sets of the form (1) (cylinder σ-algebra) such that

$$\mathbb{P}\left(C_{u_1, u_2, \dots, u_k}\right) = \frac{1}{2^k}. \tag{2}$$

(2) Compute the probabilities of the previous events A, B_n, C, D_n.

Solution:

(1) We have

$$B_n = C_{\underbrace{1,1,\ldots,1}_{n-1 \text{ times}},0}, \quad A = C_{1,1} \cup C_{0,1}, \quad C = C_1 \cap C_{1,1} \cap C_{1,1,1} \cap \cdots$$

and

$$D_n = (C_{1,1,\ldots,1} \cup C_{0,1,\ldots,1} \cup C_{1,0,1,\ldots,1} \cup \cdots \cup C_{1,1,\ldots,0})^c = {C_{1,1,\ldots,1}}^c \cap {C_{0,1,\ldots,1}}^c \cap {C_{1,0,1,\ldots,1}}^c \cap \cdots \cap {C_{1,1,\ldots,0}}^c.$$

(2) First of all, all these events are in the cylinder σ-field by the first question. We have $\mathbb{P}(B_n) = 1/2^n$ by definition of $\mathbb{P}$, $\mathbb{P}(A) = \mathbb{P}(C_{0,1}) + \mathbb{P}(C_{1,0}) = 1/4 + 1/4 = 1/2$. To calculate $\mathbb{P}(C)$, enter $C_n = C_{1,1,\ldots,1}$ where 0 appears n times. Then $C = \cap_{n \geq 0} C_n$ and the events C_n are decreasing in n, so (decreasing limit property)

$$\mathbb{P}(C) = \lim_{n\to\infty} \mathbb{P}(C_n) = \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

To calculate $\mathbb{P}(D_n)$, given the expression for D_n, it's natural to use the complementary event:

$$\mathbb{P}(D_n) = 1 - \mathbb{P}({D_n}^c) = 1 - (\mathbb{P}(C_{1,1,\ldots,1}) + \mathbb{P}(C_{0,1,\ldots,1}) + \mathbb{P}(C_{1,0,1,\ldots,1}) + \cdots \mathbb{P}(C_{1,1,\ldots,0})),$$

which is therefore $1 - \frac{n+1}{2^n}$.

□

Exercise 4. Let $(A_n)_{n\geq 1}$ be a sequence of independent events on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that

$$\mathbb{P}\left(\bigcap_{n\geq 1} A_n\right) = \prod_{n\geq 1} \mathbb{P}(A_n).$$

Solution:

Write

$$\mathbb{P}\left(\bigcap_{n\geq 1} A_n\right) = \lim_{N\to\infty} \mathbb{P}\left(\bigcap_{n=1}^{N} A_n\right) = \lim_{N\to\infty} \prod_{n=1}^{N} \mathbb{P}(A_n) = \prod_{n\geq 1} \mathbb{P}(A_n).$$

The first equality comes from the fact that the sequence of events $\bigcap_{n=1}^{N} A_n$ is decreasing in N, the second equality comes from the fact that $A_1, \ldots, A_N$ are independant, and the last equality comes from the definition of the infinite product. □

Exercise 5. Let $(\mathcal{F}_n)$ be a sequence of independent σ-fields and consider a bijection $\sigma\colon \{1,2,3,\dots\} \to \{1,2,3,\dots\}$. Show that $(\mathcal{F}_{\sigma(n)})$ is still a sequence of independent σ-fields.

Solution:
Let us fix $n \geq 1$ and let $\{i_1,\dots,i_k\}$ be an arbitrary subset of $\{1,\dots,n\}$. Take arbitrary $A_{i_1} \in \mathcal{F}_{\sigma(i_1)},\dots,A_{i_k} \in \mathcal{F}_{\sigma(i_k)}$. Let m the biggest element of $\{\sigma(1),\dots,\sigma(n)\}$. Since the sequence $(\mathcal{F}_n)$ is independent we know that $\mathcal{F}_1,\dots,\mathcal{F}_m$ are independent. In particular, given that $\{\sigma(i_1),\dots,\sigma(i_k)\} \subset \{1,\dots,m\}$, we have that

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k}).$$

which shows the independence of $\mathcal{F}_{\sigma(i_1)},\dots,\mathcal{F}_{\sigma(i_k)}$ and therefore the independence of $(\mathcal{F}_{\sigma(n)})$. $\square$

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. Fix $\alpha > 0$, $a \in \{0,1\}^k$ and let $k_* = a_1 + \cdots + a_k$. Now consider a sequence of independent events (A_n) with $\mathbb{P}(A_n) = 1/n^\alpha$ for all $n \in \mathbb{N}$ and let

$$N = \#\{n \in \mathbb{N}\colon (1_{A_n}, 1_{A_{n+1}}, \dots, 1_{A_{n+k-1}}) = a\}.$$

If $\alpha k_* > 1$ show that $N < \infty$ almost surely. If $\alpha k_* \leq 1$ show that $N = \infty$ almost surely.

Solution:
Let us define for every $n \in \mathbb{N}$ the event

$$B_n = \{(1_{A_n}, 1_{A_{n+1}}, \dots, 1_{A_{n+k-1}}) = a\}.$$

Notice that by independence of (A_i), we have

$$\mathbb{P}(B_n) = \mathbb{P}(1_{A_n} = a_1) \cdots \mathbb{P}(1_{A_{n+k-1}} = a_k).$$

We know that if $a_i = 1$, then $\mathbb{P}(1_{A_{n+i-1}} = a_i) = \mathbb{P}(A_{n+i-1}) = 1/(n+i-1)^\alpha \leq 1/n^\alpha$. On the other hand, if $a_i = 0$ we can use the trivial bound $\mathbb{P}(1_{A_{n+i-1}} = a_i) \leq 1$. Therefore $\mathbb{P}(B_n) \leq 1/n^{\alpha k_*}$. Therefore, if we have that $\alpha k_* > 1$, then $\sum_{n\geq 1} \mathbb{P}(B_n) < \infty$. By the first Borel-Cantelli Lemma, (B_n) occurs only a finite number of times almost surely, and thus $N < \infty$ almost surely.

For the second part of the exercise, let us define for $n \in \mathbb{N}$

$$B_n = \{(1_{A_{kn+1}}, 1_{A_{kn}}, \dots, 1_{A_{k(n+1)}}) = a\}.$$

By independence of (A_i), we have that (B_i) is a sequence of independent events and similarly as before, for $n > 1$ we have $\mathbb{P}(B_n) \geq 1/(k(n+1))^{\alpha k_*}(1 - 1/2^\alpha)^{k-k_*}$. The second part of the bound comes from the fact that for $n > 1$, $kn + 1 + j \geq 2$ for every $j = 0, 1, \ldots, k-1$. Hence,

$$\sum_{n \geq 1} \mathbb{P}(B_n) \geq \left(1 - \frac{1}{2^\alpha}\right)^{k-k_*} \frac{1}{k^{\alpha k_*}} \sum_{n>1} \frac{1}{(n+1)^{\alpha k_*}} .$$

If $\alpha k_* \leq 1$, we have that the series on the right hand side above is divergent, and by the second Borel-Cantelli Lemma, we conclude that B_n happens infinitely often almost surely. Therefore $N = \infty$ almost surely. □

Exercise 7. **(Diophantine approximation and Borel-Cantelli)** We denote by λ the Lebesgue measure and work on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$.

(1) Let $\epsilon > 0$ be fixed. Show that

$$\lambda\left(\left\{x \in [0,1] : \exists \text{ an } \textbf{infinite} \text{ number of rationals } p/q \text{ with } \gcd(p,q) = 1 \text{ s.t. } \left|x - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}}\right\}\right) = 0.$$

Thus, almost all x are "badly approximated by rationals at order $2+\epsilon$".

Indication. For any $q \geq 1$, consider

$$A_q := [0,1] \cap \bigcup_{p=0}^{q} \left[\frac{p}{q} - \frac{1}{q^{2+\epsilon}}, \frac{p}{q} + \frac{1}{q^{2+\epsilon}}\right].$$

(2) Show that

$$\lambda\left(\left\{x \in [0,1] : \exists \text{ an } \textbf{infinite} \text{ number of rationals } p/q \text{ with } \gcd(p,q) = 1 \text{ s.t. } \left|x - \frac{p}{q}\right| \leq \frac{1}{q^{2}}\right\}\right) = 1.$$

Thus, almost all x are "well approximated by rationals at order 2".

Solution:

(1) For all $q \geq 1$, we set

$$A_q := [0,1] \cap \bigcup_{p=0}^{q} \left[\frac{p}{q} - \frac{1}{q^{2+\epsilon}}, \frac{p}{q} + \frac{1}{q^{2+\epsilon}}\right].$$

Thus $\lambda\left(A_q\right) \leq 2/q^{1+\epsilon}$. Consequently,

$$\sum_{q \geq 1} \mathbb{P}\left(A_q\right) < +\infty.$$

By the First Borel-Cantelli lemma, $\mathbb{P}\left(\limsup_{q\to\infty} A_q\right) = 0$, so the set $\limsup_{q\to\infty} A_q$ contains the set of all reel numbers which are well approximated by rationals of order $2+\epsilon$. See http://en.wikipedia.org/wiki/Thue-Siegel-Roth_theorem

(2) We shall actually show that any irrational number can be approximated by rationals at order 2. Indeed, the set of irrational numbers has probability 1, since its complementary is countable and

therefore has probability 0 for the Lebesgue measure, this concludes.

First, we prove the following result:

Dirichlet's theorem. Let α be a real number. For any integer $N \geq 2$, there exist two integers p and q, with $0 < q < N$ such that $|q\alpha - p| < \frac{1}{N}$.

(in this statement, we do not necessarily have $\gcd(p,q) = 1$)

To do this, denote by $\{x\}$ the fractional part of x and $\lfloor x \rfloor$ the integer part of x, consider the $N+1$ numbers $0, 1, \{\alpha\}, \{2\alpha\}, \ldots, \{(N-1)\alpha\}$ and the N intervals $[0, 1/N], [1/N, 2/N], \ldots, [(N-1)/N, 1]$. According to the pigeon-hole principle, there are two of these $N+1$ numbers belonging to the same drawer.

If one of these numbers is 1 and the other $\{m\alpha\}$ with $0 \leq m \leq N-1$, then

$$|m\alpha - (\lfloor m\alpha \rfloor + 1)| = |\{m\alpha\} - 1| < \frac{1}{N}.$$

We can therefore take $q = m$ and $p = \lfloor m\alpha \rfloor + 1$.

Otherwise, we can find $0 \leq \ell < m \leq N-1$ such that $|\{m\alpha\} - \{\ell\alpha\}| < 1/N$. So

$$|(m-\ell)\alpha - (\lfloor m\alpha \rfloor - \lfloor \ell\alpha \rfloor)| = |(m\alpha - \lfloor m\alpha \rfloor) - (\ell\alpha - \lfloor \ell\alpha \rfloor)| = |\{m\alpha\} - \{\ell\alpha\}| < \frac{1}{N},$$

and we can take $q = m - \ell$ and $p = \lfloor m\alpha \rfloor - \lfloor \ell\alpha \rfloor$(we have $0 < q < N$).

In particular, we have

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN} < \frac{1}{q^2}.$$

□

This result implies that if α is irrational, there is an infinite number of rationals p/q with $\gcd(p,q) = 1$ such that $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}$.

Now assume that α is irrational. According to Dirichlet's theorem, there are infinitely many integers p, q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN} < \frac{1}{q^2}.$$

Let's assume for the sake of contradiction that the fractions $\frac{p}{q}$ take a finite number of values in irreducible form. By extraction, there exists a pair (p,q) and sequence (p_n, q_n) with $q_n \to \infty$ such that $\frac{p_n}{q_n} = \frac{p}{q}$ for all $n \geq 1$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q_n^2}.$$

Passing to the limit, we find $\alpha = \frac{p}{q}$, absurd.

□

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 8. The names of 100 mathematicians are placed in 100 wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the mathematicians are led into the room; each may look in at most 50 boxes, but must leave the room exactly as she found it and is permitted no further communication with the others. The mathematicians have a chance to plot their strategy in advance, and they are going to need it, because unless every single mathematician finds her own name all will subsequently lose their funding. Find a strategy for them which has probability of success (mathematics survive) exceeding 30%.

Remark. If each mathematician examines a random set of 50 boxes, their probability of success is $\frac{1}{2^{100}}$ (each mathematician that opens 50 boxes at random among 100 has a probability $\frac{1}{2}$ to find her name), which is very very small.

Solution:
See https://en.wikipedia.org/wiki/100_prisoners_problem. □