Week 3: Random variables, independence

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 9/10/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(X_i)_{i\geq 1} : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ be random variables. Show that

$$\sigma(X_i : i \geq 1) = \sigma\left(\bigcup_{k=1}^{\infty} \sigma(X_1, \dots, X_k)\right).$$

Recall that $\sigma(X_i : i \geq 1)$ is the smallest σ-field on Ω for which all the X_i's are measurable and $\sigma(X_1, \dots, X_k)$ is the smallest σ-field on Ω for which $X_1, \dots, X_k$ are measurable.

2 Training exercises

Exercise 2. (computing laws)

(1) Let U be a random variable following the uniform distribution on $[0,1]$. Compute and plot the cumulative distribution function of $\frac{U}{1+U}$.

(2) An integer random variable X is said to follow the Poisson distribution of parameter $a > 0$ if $\mathbb{P}(X = k) = e^{-a}\frac{a^k}{k!}$ for every integer $k \geq 0$. Consider two independent random variables X and Y which follow respectively Poisson distributions of parameters $a > 0$ and $b > 0$. Show that $X + Y$ follows a Poisson distribution and identify its parameter.

(3) An integer random variable X is said to follow the geometric distribution of parameter $p \in [0,1]$ if $\mathbb{P}(X = k) = p(1-p)^{k-1}$ for every integer $k \geq 1$. Consider two independent random variables X and Y which follow respectively geometric distributions of parameters $p \in [0,1]$ and $q \in [0,1]$. Show that $\min(X, Y)$ follows a geometric distribution and identify its parameter.

Exercise 3. Let (X_n) be a sequence of independent random variables that are all uniformly distributed on $[0,1]$. Show that

$$\text{almost surely,} \qquad \limsup_{n\to\infty} \frac{1}{\ln(n)} \ln\left(\frac{1}{X_n}\right) = 1$$

Hint. Use the Borel-Cantelli lemmas.

Exercise 4. (Function-valued random variables) Equip the set $E = \mathbb{R}^{[0,1]}$ of real-valued functions defined on $[0,1]$ with the product σ-field. Let U be a uniform random variable on $[0,1]$. Let $X = (X_t)_{0 \leq t \leq 1}$ and $Y = (Y_t)_{0 \leq t \leq 1}$ be defined as follows: $\forall t \in [0,1], \qquad X_t = 0, \qquad Y_t = \mathbb{1}_{t-U \in \mathbb{Q}}$.

(1) Show that X and Y are E-valued random variables. Do they have the same law?

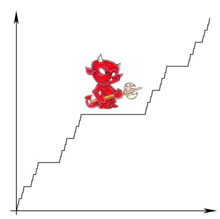
Hint. recall that cylinder sets are a generating π-system of the product σ-field.

(2) Are the following two properties true: "almost surely, for every $t \in [0,1]$, $X_t = Y_t$"? "For every $t \in [0,1]$, almost surely, $X_t = Y_t$"?

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. One can use that if $\mathcal{O}(\mathbb{R}^2)$ denotes the set of open subsets of $\mathbb{R}^2$, we have: $\mathcal{O}(\mathbb{R}^2) = \{\bigcup_{i \in I} U_i \times V_i ; U_i, V_i$ open sets of $\mathbb{R}, I$ countable.$\}$

Exercise 6. (The Devil's staircase) Construct recursively a sequence $(f_n)_{n \geq 0}$ of continuous functions on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$ as follows. First set $f_0(x) = x$ for $x \in [0,1]$. Then construct f_{n+1} from f_n by replacing f_n, on every maximal interval $[u,v]$ where it is not constant, by the piecewise linear function equal to $(f_n(u) + f_n(v))/2$ on $\left[\frac{2u}{3} + \frac{v}{3}, \frac{2v}{3} + \frac{u}{3}\right]$.



The graph of f_{devil}.

(1) Check that $|f_{n+1}(x) - f_n(x)| \leq 2^{-n}$ for every $n \geq 0$ and $x \in [0,1]$. Deduce that f_n converges uniformly on $[0,1]$ to a continuous function denoted by f_{devil}.

Let μ_{devil} be the probability measure on $[0,1]$ defined as the Stieltjes measure associated to f_{devil}.

(2) Does μ_{devil} have atoms?

We say that a measure μ on $(E, \mathcal{E})$ is *supported* on $S \in \mathcal{E}$ if $\mu(A) = \mu(A \cap S)$ (or, equivalently, $\mu(E \backslash S) = 0$). We say that two measures μ, ν on $(E, \mathcal{E})$ are *singular* if there exists $S \in \mathcal{E}$ such that μ is supported on S and ν on S^c.

(3) Show that μ_{devil} and the Lebesgue measure are singular.

Exercise 7. Let $(E, \mathcal{A})$ be a measurable space, (X, d) a metric space and $(f_n : (E, \mathcal{A}) \to (X, \mathcal{B}(X)))_{n \geq 1}$ a sequence of measurable functions. Assume that $(f_n)_{n \geq 1}$ converges pointwise to a function $f : E \to X$ (that is, for every $x \in E$, $f_n(x)$ converges to $f(x)$ as $n \to \infty$.

Show that $f : (E, \mathcal{A}) \to (X, \mathcal{B}(X))$ is measurable.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 8. You are handed two envelopes, and you know that each contains a positive integer CHF amount and that the two amounts are different (the values of these two amounts are modeled as constants that are unknown). Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. Is there a strategy that has a probability strictly larger than 1/2 of ending up with the envelope with the larger amount?