Week 3: Random variables, independence

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 9/10/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(X_i)_{i\geq 1} : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ be random variables. Show that

$$\sigma(X_i : i \geq 1) = \sigma\left(\bigcup_{k=1}^{\infty} \sigma(X_1, \ldots, X_k)\right).$$

Recall that $\sigma(X_i : i \geq 1)$ is the smallest σ-field on Ω for which all the X_i's are measurable and $\sigma(X_1, \ldots, X_k)$ is the smallest σ-field on Ω for which $X_1, \ldots, X_k$ are measurable.

Solution:

We arguy by double inclusion.

We first show that

$$\sigma\left(\bigcup_{k=1}^{\infty} \sigma(X_1, \ldots, X_k)\right) \subset \sigma(X_i : i \geq 1).$$

To this end, for fixed $k \geq 1$, we clearly have $\sigma(X_1, \ldots, X_k) \subset \sigma(X_i : i \geq 1)$. Hence

$$\bigcup_{k=1}^{\infty} \sigma(X_1, \ldots, X_k) \subset \sigma(X_i : i \geq 1)$$

which implies

$$\sigma\left(\bigcup_{k=1}^{\infty} \sigma(X_1, \ldots, X_k)\right) \subset \sigma(X_i : i \geq 1).$$

Second, we show that

$$\sigma(X_i : i \geq 1) \subset \sigma\left(\bigcup_{k=1}^{\infty} \sigma(X_1, \ldots, X_k)\right).$$

To this end, we show that for every fixed $i \geq 1$, X_i is measurable when Ω is equipped with the σ field

$\sigma\left(\bigcup_{k=1}^{\infty}\sigma(X_1,\ldots,X_k)\right)$. To this end, take $A\in\mathcal{E}$. Then $X_i^{-1}(A)\in\sigma(X_i)$ by definition of $\sigma(X_i)$. But

$$\sigma(X_i)\subset\sigma(X_1,\ldots,X_i)\subset\bigcup_{k=1}^{\infty}\sigma(X_1,\ldots,X_k)\subset\sigma\left(\bigcup_{k=1}^{\infty}\sigma(X_1,\ldots,X_k)\right).$$

□

2 Training exercises

Exercise 2. (computing laws)

(1) Let U be a random variable following the uniform distribution on $[0,1]$. Compute and plot the cumulative distribution function of $\frac{U}{1+U}$.

(2) An integer random variable X is said to follow the Poisson distribution of parameter $a>0$ if $\mathbb{P}(X=k)=e^{-a}\frac{a^k}{k!}$ for every integer $k\geq 0$. Consider two independent random variables X and Y which follow respectively Poisson distributions of parameters $a>0$ and $b>0$. Show that $X+Y$ follows a Poisson distribution and identify its parameter.

(3) An integer random variable X is said to follow the geometric distribution of parameter $p\in[0,1]$ if $\mathbb{P}(X=k)=p(1-p)^{k-1}$ for every integer $k\geq 1$. Consider two independent random variables X and Y which follow respectively geometric distributions of parameters $p\in[0,1]$ and $q\in[0,1]$. Show that $\min(X,Y)$ follows a geometric distribution and identify its parameter.

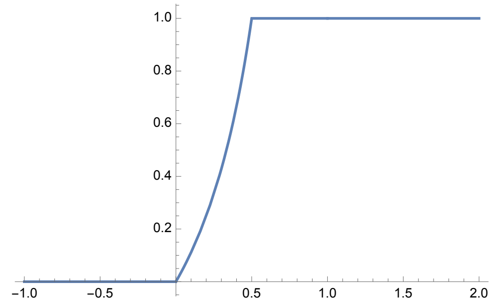
Solution:

(1) Set $F(x)=\mathbb{P}(U/(1+U)\leq x)$. Since $\mathbb{P}(U\in[0,1])=1$, we have $F(x)=0$ for $x\leq 0$ and $F(x)=1$ for $x\geq 1$. For $x\in[0,1]$ write

$$\frac{U}{1+U}\leq x \iff U\leq x+Ux \iff U\leq\frac{x}{1-x}.$$

Observe that $x//(1-x)\leq 1$ if and only if $x\leq 1/2$. We conclude that

$$F(x)=\begin{cases}0 & \text{for } x\leq 0\\ \frac{x}{1-x} & \text{for } 0\leq x\leq 1/2\\ 1 & \text{for } x\geq 1/2\end{cases}$$



(2) For $n \geq 0$ we have

$$\mathbb{P}(X+Y=n)=\sum_{k=0}^{n}\mathbb{P}(X=k,Y=n-k)=\sum_{k=0}^{n}\mathbb{P}(X=k)\mathbb{P}(Y=n-k)$$

where for the second equality we used the independence of X and Y. We now use the distribution of X and Y together with the Binomial theorem to obtain

$$\begin{aligned}\mathbb{P}(X+Y=n)&=\sum_{k=0}^{n}e^{-a}\frac{a^k}{k!}\cdot e^{-b}\frac{b^{n-k}}{(n-k)!}=e^{-(a+b)}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\\&=e^{-(a+b)}\frac{(a+b)^n}{n!}.\end{aligned}$$

This shows that $X+Y$ follows the Poisson distribution of parameter $a+b$.

(3) Set $Z=\min(X,Y)$. The idea is to write, for $k \geq 1$,

$$\mathbb{P}(Z=k)=\mathbb{P}(Z\geq k)-\mathbb{P}(Z\geq k+1).$$

Indeed, events of the form $\{Z \geq k\}$ are simpler to manipulate when a minimum is involved in the definition of Z, since we can write:

$$\mathbb{P}(Z\geq k)=\mathbb{P}(\min(X,Y)\geq k)=\mathbb{P}(X\geq k,Y\geq k)=\mathbb{P}(X\geq k)\mathbb{P}(Y\geq k)$$

by independence. Then compute

$$\mathbb{P}(X\geq k)=\sum_{i=k}^{\infty}p(1-p)^i=p(1-p)^k\sum_{i=0}^{\infty}(1-p)^i=p(1-p)^k\cdot\frac{1}{1-(1-p)}=(1-p)^k.$$

Thus $\mathbb{P}(Z\geq k)=((1-p)(1-q))^k$ and

$$\mathbb{P}(Z=k)=((1-p)(1-q))^k-((1-p)(1-q))^{k+1}=(1-(1-p)(1-q))((1-p)(1-q))^k.$$

We conclude that Z follows the geometric distribution of parameter $1-(1-p)(1-q)$.

□

Exercise 3. Let (X_n) be a sequence of independent random variables that are all uniformly distributed on $[0,1]$. Show that

$$\text{almost surely,} \qquad \limsup_{n\to\infty} \frac{1}{\ln(n)} \ln\left(\frac{1}{X_n}\right) = 1$$

Hint. Use the Borel-Cantelli lemmas.

Solution:

For $c > 0$ we let $A_c^n = \{\log(1/X_n)/\log n \geq c\} = \{X_n \leq 1/n^c\}$. Then

$$\mathbb{P}(A_c^n) = \mathbb{P}(X_n \leq 1/n^c) = \frac{1}{n^c}.$$

Thus, for $c > 1$ we have $\sum_{n\geq 1} \mathbb{P}(A_c^n) < \infty$. Hence $\limsup_{n\to\infty} \log(1/X_n)/\log n \leq c$ almost surely by the first Borel-Cantelli lemma. On the other hand, for $c \in (0,1]$ we have $\sum_{n\geq 1} \mathbb{P}(A_c^n) = \infty$. Hence $\limsup_{n\to\infty} \log(1/X_n)/\log n \geq c$ almost surely by the independence of the (X_n) and the second Borel-Cantelli lemma. The claim in the question now follows since

$$\left\{\limsup_{n\to\infty} \log(1/X_n)/\log n = 1\right\} = \bigcap_{k\geq 1} \left\{\limsup_{n\to\infty} \log(1/X_n)/\log n \leq 1 + 1/k\right\}$$
$$\cap \bigcap_{k>1} \left\{\limsup_{n\to\infty} \ln(\frac{1}{X_n}/\log n \geq 1 - 1/k\right\}$$

and the countable intersection of events of probability 1 again has probability 1. □

Exercise 4. (Function-valued random variables) Equip the set $E = \mathbb{R}^{[0,1]}$ of real-valued functions defined on $[0,1]$ with the product σ-field. Let U be a uniform random variable on $[0,1]$. Let $X = (X_t)_{0\leq t\leq 1}$ and $Y = (Y_t)_{0\leq t\leq 1}$ be defined as follows: $\forall t \in [0,1], \qquad X_t = 0, \qquad Y_t = \mathbb{1}_{t-U\in\mathbb{Q}}$.

(1) Show that X and Y are E-valued random variables. Do they have the same law?

Hint. recall that cylinder sets are a generating π-system of the product σ-field.

(2) Are the following two properties true: "almost surely, for every $t \in [0,1]$, $X_t = Y_t$"? "For every $t \in [0,1]$, almost surely, $X_t = Y_t$"?

Solution:

(1) We have seen in the lecture that a function taking values in a product space equiped with the product σ-field is measurable if and only if all the projections are measurable. In other words, $(X_t)_{0\leq t\leq 1}$ and $(Y_t)_{0\leq t\leq}$ are random variables if and only if for every $0 \leq t \leq 1$, X_t and Y_t are random variables, which is clearly the case.

Yes, they have the same law! Since E is equipped with the product σ-field, it is enough to check that X and Y have the same finite dimensional distributions.For $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$, $(X_{t_1}, \ldots, X_{t_n})$ and $(Y_{t_1}, \ldots, Y_{t_n})$ are both equal almost surely to $(0, 0, \ldots, 0)$, so they indeed have the same distribution.

(2) The first one is not true. Assume by contradiction that there exists an event A with $\mathbb{P}(A) = 1$ on which for every $t \in [0,1]$, $X_t = Y_t$. Take $\omega \in A$. But then $X_{U(\omega)}(\omega) = 0$ and $Y_{U(\omega)}(\omega) = 1$. A contradiction.

The second one is true: for fixed $t \in [0,1]$, we have $X_t = Y_t = 0$ almost surely.

□

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. One can use that if $\mathcal{O}(\mathbb{R}^2)$ denotes the set of open subsets of $\mathbb{R}^2$, we have: $\mathcal{O}(\mathbb{R}^2) = \{\bigcup_{i \in I} U_i \times V_i;\, U_i, V_i$ open sets of $\mathbb{R}, I$ countable.$\}$

Solution:

We check the double inclusion.

– We first show that $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R}^2)$. For open sets O, O' in $\mathbb{R}$, $O \times O'$ is open in $\mathbb{R}^2$. Next, consider G_1 defined by

$$G_1 = \{A \in \mathcal{B}(\mathbb{R});\, A \times B \in \mathcal{B}(\mathbb{R}^2) \text{ for every open set } B \text{ in } \mathbb{R}\}.$$

It is a simple matter to check that G_1 is a σ-field. Since G_1 contains open sets of $\mathbb{R}$, $G_1 = \mathcal{B}(\mathbb{R})$. Next, consider G_2 defined by:

$$G_2 = \{B \in \mathcal{B}(\mathbb{R});\, A \times B \in \mathcal{B}(\mathbb{R}^2) \text{ for every } A \in \mathcal{B}(\mathbb{R})\}.$$

It is a simple matter to check that G_2 is a σ-field. By what precedes, G_2 contains open sets of $\mathbb{R}$, so $G_2 = \mathcal{B}(\mathbb{R})$. Therefore $\mathcal{F} \times \mathcal{G} \in \mathcal{B}(\mathbb{R}^2)$ for $\mathcal{F}, \mathcal{G} \in \mathcal{B}(\mathbb{R})$, so $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R}^2)$. Indeed, recall that by definition $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{F} \times \mathcal{G}; \mathcal{F}, \mathcal{G} \in \mathcal{B}(\mathbb{R}))$.

— Conversely, let us show that $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. By the result given in the statement of the exercise, since $\mathcal{B}(\mathbb{R}^2) = \sigma(\mathcal{O}(\mathbb{R}^2))$, it suffices to show that if I is countable and U_i, V_i $(i \in \mathbb{R})$ are open sets of $\mathbb{R}$, then

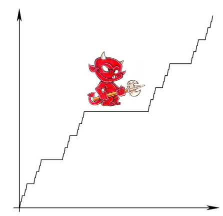
$$\bigcup_{i \in I} U_i \times V_i \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}).$$

This is clearly the case, since $U_i, V_i \in \mathcal{B}(\mathbb{R})$ and I is **countable**.

Remark. This proof shows that $\mathcal{B}(X \times Y) = \mathcal{B}(X) \otimes \mathcal{B}(Y)$ when X and Y are two metric spaces with a countable basis of open sets (or, equivalently, if they are separable, meaning that they have a countable dense sequence).

□

Exercise 6. (The Devil's Staircase) Construct recursively a sequence $(f_n)_{n\geq 0}$ of continuous functions on $[0,1]$ such that $f(0)=0$ and $f(1)=1$ as follows. First set $f_0(x)=x$ for $x\in[0,1]$. Then construct f_{n+1} from f_n by replacing f_n, on every maximal interval $[u,v]$ where it is not constant, by the piecewise linear function equal to $(f_n(u)+f_n(v))/2$ on $\left[\frac{2u}{3}+\frac{v}{3},\frac{2v}{3}+\frac{u}{3}\right]$.



The graph of f_{devil}.

(1) Check that $|f_{n+1}(x)-f_n(x)|\leq 2^{-n}$ for every $n\geq 0$ and $x\in[0,1]$. Deduce that f_n converges uniformly on $[0,1]$ to a continuous function denoted by f_{devil}.

Let μ_{devil} be the probability measure on $[0,1]$ defined as the Stieltjes measure associated to f_{devil}.

(2) Does μ_{devil} have atoms?

We say that a measure μ on $(E,\mathcal{E})$ is *supported* on $S\in\mathcal{E}$ if $\mu(A)=\mu(A\cap S)$ (or, equivalently, $\mu(E\backslash S)=0$). We say that two measures μ,ν on $(E,\mathcal{E})$ are *singular* if there exists $S\in\mathcal{E}$ such that μ is supported on S and ν on S^c.

(3) Show that μ_{devil} and the Lebesgue measure are singular.

Solution:

(1) It is a simple matter to check that $|f_{n+1}(x)-f_n(x)|\leq 2^{-n}$ for every $n\geq 0$ and $x\in[0,1]$ by using the definition of f_n. Il follows that the series of functions $\sum_{n\geq 0}(f_{n+1}-f_n)$ converges uniformly to a continuous function on $[0,1]$.

(2) Since f_{devil} is continuous, μ_{devil} has no atoms.

(3) Let us construct a set K with 0 Lebesgue measure which supports μ_{devil}. First set $K_0=[0,1]$. Then, by induction, once K_n, which is a finite union of closed intervals, has been defined, let K_{n+1} be obtained by removing in each interval of K_n an open interval centered in the center of the latter interval, with length $1/3$ of the latter interval. Then set

$$K=\bigcap_{n\geq 0}K_n,$$

which is compact set (called Cantor's tryadic set).

Now, by construction, for every $N \geq 0$, $\mu_{\text{devil}}(\bigcap_{n=0}^{N} K_n) = 1$. It follows that

$$\mu_{\text{devil}}(K) = \mu_{\text{devil}}\left(\bigcap_{n=0}^{\infty} K_n\right) = 1.$$

Similarly, if λ denotes the Lebesgue measure on $[0,1]$, by construction, $\lambda(\bigcap_{n=0}^{N} K_n) = (2/3)^N$, so $\lambda(K) = 0$.

Remark. It is possible to check that

$$K = \left\{ \sum_{n=1}^{\infty} \frac{x_n}{3^n} : x_i \in \{0,2\} \text{ for } i \geq 1 \right\}.$$

We already saw that K is compact, and it is possible to show that in addition K is non countable, has empty interior, and that every point in K is an accumulation point.

Remark. To simulate a random variable X whose law is μ_{devil} one can proceed as follows: let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. uniform random variables on $\{0,2\}$. Then set

$$X = \sum_{n=1}^{\infty} \frac{X_n}{3^n}.$$

□

Exercise 7. Let $(E,\mathcal{A})$ be a measurable space, (X,d) a metric space and $(f_n : (E,\mathcal{A}) \to (X,\mathcal{B}(X))_{n\geq 1}$ a sequence of measurable functions. Assume that $(f_n)_{n\geq 1}$ converges pointwise to a function $f : E \to X$ (that is, for every $x \in E$, $f_n(x)$ converges to $f(x)$ as $n \to \infty$.

Show that $f : (E,\mathcal{A}) \to (X,\mathcal{B}(X))$ is measurable.

Solution:

It is enough to show that $f^{-1}(F)$ is measurable for every closed set F. Recall that the distance to a closed set is a 1-lipschitz function (meaning that $x \to d(x,F)$ is 1-lipschitz) and that $x \in F$ if and only if $d(x,F) = 0$. We then write:

$$\begin{aligned} f^{-1}(F) &= \{x \in X; d(f(x),F) = 0\} = \left\{x \in X;\ \lim_{n\to\infty} d(f_n(x),F) = 0\right\} \\ &= \bigcap_{p\geq 1} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \left\{x \in X; d(f_n(x),F) \leq \frac{1}{p}\right\}, \end{aligned}$$

which is measurable as a countable unions and intersections of measurable sets. Indeed, $x \to d(f_n(x),F)$ is measurable, being the composition of the measurable function f_n with the 1-lipschitz function $y \to d(y,F)$ (continuous, hence measurable).

□

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 8. You are handed two envelopes, and you know that each contains a positive integer CHF amount and that the two amounts are different (the values of these two amounts are modeled as constants that are unknown). Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. Is there a strategy that has a probability strictly larger than 1/2 of ending up with the envelope with the larger amount?

Solution:

The strategy is as follows. Let V be the number you see. You then draw a real number $\omega \in \mathbb{R}$ independently at random (according to a strictly positive probability distribution on $\mathbb{R}$ - a Gaussian distribution, for example). Finally, you compare V and ω: if $V \geq \omega$, you keep the envelope, and if not, you change it.

Why does this work? We note $a < b$ the numbers chosen by the devil. So $V = a$ or $V = b$ with probability 1/2. There are three possible outcomes:

(1) $\omega < a$. In this case, we always have $\omega < V$, and you always keep keep the envelope. If $V = a$, you lose, and if $V = b$, you win. So you win with probability 1/2.

(2) $\omega \geq b$. By the same reasoning, you win with probability 1/2.

(3) $a \leq \omega < b$. In this case, if $V = a$, then as $V \leq \omega$, you change envelopes and end up with the larger amount, and if $V = b$, then $V > \omega$, you keep the envelope and end up with the larger amount. In both cases, you win.

Thus, if we denote by $p > 0$ the probability that you choose ω in the interval $[a, b]$, you win with probability

$$(1-p)\frac{1}{2} + p = \frac{1}{2} + \frac{p}{2} > \frac{1}{2}.$$

□