Week 5: classical laws and independence

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 23/10/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let (X,Y) be a random variable with values in $\mathbb{R}^2$ whose joint distribution has the density $f_{(X,Y)}(x,y) = \frac{1}{4}(1+xy)\mathbb{1}_{-1 \le x,y \le 1}$.

(1) Find the law of X.

(2) Compute $\mathbb{E}[X\mathbb{1}_{X<1/2}]$.

(3) Compute $\mathbb{E}\left[\frac{1}{X}\right]$.

(4) Compute $\mathbb{E}[XY]$.

(5) Compute $\mathbb{P}(X \le Y)$. Is the result surprising?

(6) Are the random variables X and Y independent? Justify your answer.

2 Training exercises

Exercise 2. Let U be a uniform random variable on $[-1,1]$. Compute $\mathbb{E}[e^U]$.

Exercise 3. Let X be a real random variable that follows an exponential distribution with parameter 1. Let $\lambda > 0$. Show that λX follows an exponential distribution with parameter $1/\lambda$.

Exercise 4. Let Z be a real random variable with density $\frac{1}{\pi} \cdot \frac{1}{1+x^2}$ (it is a so-called Cauchy random variable). For which values of $\alpha \in \mathbb{Z}$ is the random variable Z^α integrable?

Exercise 5. Let X and Y be two independent random variables, where X follows an exponential distribution with parameter $\lambda > 0$, and Y follows a geometric distribution with parameter $p \in (0,1)$. Compute $\mathbb{P}(X > Y)$.

Exercise 6. Let X and Y be two independent real variables. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ be a measurable function. Show that $\mathbb{E}[F(X,Y)] = \mathbb{E}[g(Y)]$, where $g : \mathbb{R} \to \mathbb{R}$ is the function defined by $g(y) = \mathbb{E}[F(X,y)]$ for $y \in \mathbb{R}$.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 7. Let X be an exponential random variable with a parameter of 1, and $a > 0$. Does the random variable $\min(X, a)$ have a density?

Exercise 8. Let T be an exponential random variable and U an independent uniform random variable on $[0, 1]$. Set $X = \sqrt{T}\cos(2\pi U)$ and $Y = \sqrt{T}\sin(2\pi U)$. Find the law of (X, Y).

Exercise 9. Let (X_n) be a sequence of independent real random variables.

(1) Show that the radius of convergence R of the power series $\sum_{n\geq 0} X_n z^n$ is almost surely constant.

(2) Now assume that the random variables $(X_n)_{n\geq 0}$ have the same law. Show that if $\mathbb{E}[\ln(|X_1|)^+] = \infty$, then $R = 0$ almost surely, and if $\mathbb{E}[\ln(|X_1|)^+] < \infty$, then $R \geq 1$ almost surely (here $x^+ = \max(x, 0)$ represents the positive part of a real number x).

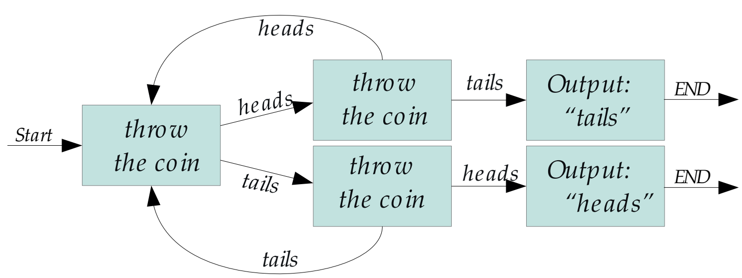
Exercise 10. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with law given by

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2, \qquad n = 0, 1, \ldots.$$

Show that with probability 1, there is no point z_0 on the unit cercle such that the power series $F(z) = \sum_{n\geq 0} X_n z^n$ can be extended in an open ball around z_0 into a function which can be expanded in a power series around z_0.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 11. We have a biased coin that comes up heads with a probability of p, and we want to use it to generate a fair coin toss. John von Neumann came up with the following algorithm:



Show that this works and compute the average number of times the coin is tossed.