Week 6: Law of large numbers

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 30/10/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $(X_n)_{n\geq 1}$ be a sequence of real random variables that almost surely converges to X. Show that $f(X_n)$ almost surely converges to $f(X)$.

(2) Let $(X_n)_{n\geq 1}$ be a sequence of real random variables that almost surely converges to X, and let $(Y_n)_{n\geq 1}$ be a sequence of real random variables that almost surely converges to Y. Show that (X_n, Y_n) almost surely converges to (X, Y).

(3) Suppose that $(U_n)_{n\geq 1}$ are i.i.d. uniform random variables on $\{-1, +1\}$ and consider $\beta > 0$. Discuss the convergence of the series $\sum_{n\geq 1} \frac{U_n}{n^\beta}$.

You may assume that the converse of the Kolmogorov three series theorem is true.

2 Training exercises

Exercise 2. Let $(Z_n, n \geq 1)$ be a sequence of random variables such that for all integers $n \geq 1$, Z_n is an exponential random variable with parameter n. Show that Z_n almost surely converges to 0 as $n \to \infty$.

Exercise 3. Let $(X_n)_{n\geq 1}$ be i.i.d. integrable random variables with the same law as X. Define

$$M_n = \frac{1}{n} \sum_{i=1}^{n} X_i X_{i+1} .$$

Show that the sequence (M_n) converges almost surely and find its (almost sure) limit.

Exercise 4. Consider a sequence $(X_n)_{n\geq 1}$ of i.i.d. integrable random variables with the same law as X and $\mathbb{E}[X] = 0$.

(1) Show that if $\mathbb{E}[X^2] < \infty$ then $\sum_{n \geq 1} X_n/n$ converges almost surely.

(2) Suppose now instead that X and $-X$ have the same law. Show that in this case the series $\sum_{n \geq 1} X_n/n$ converges almost surely as well.

Exercise 5. Suppose that $(X_n)_{n \geq 1}$ are i.i.d. random variables taking values in $(0, \infty)$ with the same law as X. Also suppose that $\mathbb{E}[|\log X|] < \infty$.

(1) Show that almost surely, as $n \to \infty$,

$$X_1 \cdots X_n = e^{\alpha n + o(n)}$$

where $\alpha = \mathbb{E}[\log X]$.

(2) Fix $a > 1$. Construct a sequence $(Y_n)_{n \geq 1}$ with values in $(0, \infty)$ of random variables such that $\mathbb{E}[Y_n] = a^n$ for all $n \geq 1$ and $Y_n \to 0$ almost surely as $n \to \infty$.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. Let $(X_n)_{n \geq 1}$ be a sequence of real random variables. Show that there exists a sequence $c_n \to \infty$ such that X_n/c_n converges almost surely to 0.

Exercise 7. Let $(X_n)_{n \geq 1}$ be an i.i.d. sequence with the same law as X such that $\mathbb{E}[X^{2p}] < \infty$ for all integers $p \geq 1$. Also assume that $\mathbb{E}[X] = 0$.

(1) Show that for all integers $p \geq 1$ there exists a constant $C_p < \infty$ such that

$$\mathbb{E}((X_1 + \cdots + X_n)^{2p}) \leq C_p \, n^p \, .$$

(2) Deduce that $(X_1 + \cdots + X_n)/n^{1/2+\delta} \to 0$ as $n \to \infty$ almost surely for all $\delta > 0$.

Exercise 8. Find an integrable random variable X with $\mathbb{E}[X] = 0$ such that if $(X_n)_{n \geq 1}$ is a sequence of i.i.d. random variables with the same law as X then the series $\sum_{n \geq 1} X_n/n$ does not converge with positive probability.

You may assume that the converse of the Kolmogorov three series theorem is true.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9. We color 10% of a sphere in blue and the rest in red. Show that it is possible to inscribe a cube in the sphere with all its vertices being red.