Week 6: Law of large numbers

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 30/10/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $(X_n)_{n\geq 1}$ be a sequence of real random variables that almost surely converges to X. Show that $f(X_n)$ almost surely converges to $f(X)$.

(2) Let $(X_n)_{n\geq 1}$ be a sequence of real random variables that almost surely converges to X, and let $(Y_n)_{n\geq 1}$ be a sequence of real random variables that almost surely converges to Y. Show that (X_n, Y_n) almost surely converges to (X, Y).

(3) Suppose that $(U_n)_{n\geq 1}$ are i.i.d. uniform random variables on $\{-1, +1\}$ and consider $\beta > 0$. Discuss the convergence of the series $\sum_{n\geq 1} \frac{U_n}{n^\beta}$.

You may assume that the converse of the Kolmogorov three series theorem is true.

Solution:

(1) We have the inclusion of events:

$$\{X_n \to X\} \subset \{f(X_n) \to f(X)\}.$$

Here we have used "probabilistic notation"; recall that this means that

$$\{\omega \in \Omega : X_n(\omega) \to X(\omega)\} \subset \{\omega \in \Omega : f(X_n(\omega)) \to f(X(\omega))\}.$$

Since the event on the left has probability 1, we can conclude that the one on the right also has probability 1.

Note: It's important to note that these two events are not necessarily equal (for example, if f is a constant function).

(2) We have the equality of events

$$\{X_n \to X\} \cap \{Y_n \to Y\} = \{(X_n, Y_n) \to (X, Y)\}.$$

Since the left-hand event has probability 1 (as it's the intersection of two events with probability 1), we can conclude that the right-hand event also has probability 1.

(3) By the three series criterion, the sum converges almost surely if the following three conditions are satisfied and does not converge almost surely otherwise for $a > 0$:

(a) $\sum_{n\geq 1} \mathbb{P}\left(|U_n n^{-\beta}| > a\right) < \infty$,

(b) $\sum_{n\geq 1} \mathbb{E}\left[U_n n^{-\beta} \mathbb{1}_{|U_n n^{-\beta}|<a}\right] < \infty$

(c) $\sum_{n\geq 1} \mathsf{Var}(U_n n^{-\beta} \mathbb{1}_{|U_n n^{-\beta}|<a}) < \infty$.

Note that (i) and (ii) are clearly satisfied. For (iii), we see from the definition that

$$\sum_{n\geq 1} \mathsf{Var}(U_n n^{-\beta} \mathbb{1}_{|U_n n^{-\beta}|<a}) = \sum_{n\geq 1} n^{-2\beta} \mathbb{1}_{n^{-\beta}<a}$$

Observe that $n^{-\beta} < a$ if and only if $n > (1/a)^{1/\beta}$. Thus

$$\sum_{n\geq 1} \mathsf{Var}(U_n n^{-\beta} \mathbb{1}_{|U_n n^{-\beta}|<a}) = \sum_{n\geq (1/a)^{1/\beta}}^{\infty} n^{-2\beta}$$

This sum converges if and only if $\beta > 1/2$. We therefore deduce that if $\beta > 1/2$ then the series $\sum_{n\geq 1} U_n n^{-\beta}$ converges almost surely and if $\beta \leq 1/2$ the series $\sum_{n\geq 1} U_n n^{-\beta}$ does not converge almost surely.

Remark. It is interesting to note that for $1/2 < \beta \leq 1$ almost surely the series $\sum_{n\geq 1} \frac{U_n}{n^\beta}$ is convergent, but not absolutely convergent.

Remark. When $x > 0$ is a real number, we use the notation $\sum_{n\geq x} a_n$ for $\sum_{n=\lceil x\rceil}^{\infty} a_n$.

□

2 Training exercises

Exercise 2. Let $(Z_n, n \geq 1)$ be a sequence of random variables such that for all integers $n \geq 1$, Z_n is an exponential random variable with parameter n. Show that Z_n almost surely converges to 0 as $n \to \infty$.

Solution:

Let $\epsilon > 0$. We have $\mathbb{P}(Z_n > \epsilon) = e^{-n\epsilon}$. Therefore,

$$\sum_{n\geq 1} \mathbb{P}(Z_n > \epsilon) < \infty.$$

By the result seen in the lecture, this implies that $Z_n \to 0$ almost surely.

Let us remind the argument. According to the (first) Borel-Cantelli lemma, for any $\epsilon > 0$, almost surely, for every n sufficiently large, we have $Z_n \leq \epsilon$. Therefore, almost surely, for any integer $k \geq 1$, for n sufficiently large, we have $0 \leq Z_n \leq 1/k$. It follows that Z_n almost surely converges to 0.
WARNING. An interchange of "for any $\epsilon > 0$" and "almost surely" has been performed here, which is made possible by considering a countable sequence (in general it is not possible to interchange "$\forall$ on an uncountable set" and "a.s."). □

Exercise 3. Let $(X_n)_{n\geq 1}$ be i.i.d. integrable random variables with the same law as X. Define

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i X_{i+1} \,.$$

Show that the sequence (M_n) converges almost surely and find its (almost sure) limit.

Solution:
We let

$$M_n^+ = \frac{1}{n}\sum_{i=1}^{n} X_{2i}X_{2i+1} \quad \text{and} \quad M_n^- = \frac{1}{n}\sum_{i=1}^{n} X_{2i-1}X_{2i}\,.$$

Since the sequences $(X_{2i}X_{2i+1})_i$ and $(X_{2i-1}X_{2i})_i$ are i.i.d. and furthermore $\mathbb{E}[X_{2i}X_{2i+1}] = \mathbb{E}[X_{2i-1}X_{2i}] = \mathbb{E}[X]^2$ the strong law of large numbers implies that

$$M_n^+ \to \mathbb{E}[X]^2 \quad \text{and} \quad M_n^- \to \mathbb{E}[X]^2 \quad \text{as } n \to \infty \text{ a.s.}$$

Note that $M_{2n} = M_{n-1}^+ \cdot (1/2 - 1/(2n)) + M_n^-/2$ and $M_{2n+1} = (M_n^+ + M_n^-)\cdot n/(2n+1)$ we deduce that $M_n \to \mathbb{E}[X]^2$ as $n \to \infty$ almost surely. □

Exercise 4. Consider a sequence $(X_n)_{n\geq 1}$ of i.i.d. integrable random variables with the same law as X and $\mathbb{E}[X] = 0$.

(1) Show that if $\mathbb{E}[X^2] < \infty$ then $\sum_{n\geq 1} X_n/n$ converges almost surely.

(2) Suppose now instead that X and $-X$ have the same law. Show that in this case the series $\sum_{n\geq 1} X_n/n$ converges almost surely as well.

Solution:

(1) We use the Kolmogorov two-series theorem in the following form: if (Z_n) is a sequence of independent random variables with $\sum_{n\geq 1} \mathbb{E}[Z_n^2] < \infty$ and $\mathbb{E}[Z_n] = 0$ for all $n \geq 1$ then the series $\sum_{n\geq 1} Z_n$ converges almost surely. We apply this statement to the sequence $Z_n = X_n/n$. Indeed, this is applicable since $\mathbb{E}[Z_n] = \mathbb{E}[X]/n = 0$ and $\mathbb{E}[Z_n^2] = \mathbb{E}[X^2]/n^2$ and $\mathbb{E}[X^2] < \infty$.

(2) We use the three series criterion. Fix $a > 0$.

First of all, we have

$$\sum_{n\geq 1} \mathbb{P}(|X_n/n| > a) = \sum_{n\geq 1} \mathbb{P}(|X| > an) < \infty$$

since $\mathbb{E}[|X|] < \infty$ as seen in the lecture. Indeed, recall from Exercise sheet 4 Exercise 3.2 that for a nonnegative random variable Z we have $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z \geq t)\,\mathrm{d}t$, so that

$$\infty > \mathbb{E}\left[\frac{2|X|}{a}\right] = \int_0^\infty \mathbb{P}\left(|X| \geq \frac{a}{2}t\right)\mathrm{d}t = \sum_{n=0}^{\infty} \int_n^{n+1} \mathbb{P}\left(|X| \geq \frac{a}{2}t\right)\mathrm{d}t \geq \sum_{n=1}^{\infty} \mathbb{P}\left(|X| \geq \frac{a}{2}n\right) > \sum_{n=1}^{\infty} \mathbb{P}(|X| > an).$$

Remark. Here we gave taken $\frac{a}{2}$ instead of a because otherwise we would have obtained the inequality $\sum_{n\geq 1} \mathbb{P}(|X| \geq an) < \infty$ while we want $\sum_{n\geq 1} \mathbb{P}(|X| > an) < \infty$.

Moreover, $\mathbb{E}\left[X_n/n \mathbb{1}_{|X_n/n|<a}\right] = 0$ for all $n \geq 1$ since X_n has the same law as $-X_n$. Therefore $\sum_{n\geq 1} \mathbb{E}\left[X_n/n \mathbb{1}_{|X_n/n|<a}\right] = 0$.

Finally, we bound

$$\sum_{n\geq 1} \mathrm{Var}(X_n/n \mathbb{1}_{|X_n/n|<a}) = \sum_{n\geq 1} n^{-2}\mathbb{E}\left[X^2 \mathbb{1}_{|X|<an}\right] = \mathbb{E}\left[X^2 \sum_{n\geq 1} \frac{\mathbb{1}_{n>|X|/a}}{n^2}\right] = \mathbb{E}\left[X^2 \sum_{n\geq 1} \frac{\mathbb{1}_{n>|X|/a}}{n^2}\right]$$

Notice that there exists a constant $C > 0$ such that for every $k \geq 1$ we have

$$\sum_{n=k}^{\infty} \frac{1}{n^2} \leq \frac{C}{k}. \tag{1}$$

Indeed, for $k \geq 2$ we have

$$\sum_{n=k}^{\infty} \frac{1}{n^2} \leq \sum_{n=k}^{\infty} \int_{n-1}^{n} \frac{\mathrm{d}t}{t^2} = \int_{k-1}^{\infty} \frac{\mathrm{d}t}{t^2} = \frac{1}{k-1} \leq \frac{2}{k}.$$

We can therefore choose $C > 0$ such that (1) holds for every $k \geq 1$. In particular, for every $x > 0$:

$$\sum_{n>x}^{\infty} \frac{1}{n^2} \leq \frac{C}{\lceil x \rceil} \leq \frac{C}{x}$$

As a consequence,

$$\mathbb{E}\left[X^2 \sum_{n\geq 1} \frac{\mathbb{1}_{n>|X|/a}}{n^2}\right] \leq C\mathbb{E}\left[X^2 \frac{1}{|X|/a}\right] = aC\mathbb{E}[|X|] < \infty.$$

□

Exercise 5. Suppose that $(X_n)_{n\geq 1}$ are i.i.d. random variables taking values in $(0,\infty)$ with the same law as X. Also suppose that $\mathbb{E}[|\log X|] < \infty$.

(1) Show that almost surely, as $n \to \infty$,

$$X_1 \cdots X_n = e^{\alpha n + o(n)}$$

where $\alpha = \mathbb{E}[\log X]$.

(2) Fix $a > 1$. Construct a sequence $(Y_n)_{n\geq 1}$ with values in $(0,\infty)$ of random variables such that $\mathbb{E}[Y_n] = a^n$ for all $n \geq 1$ and $Y_n \to 0$ almost surely as $n \to \infty$.

Solution:

(1) Recall that $f(n) = o(g(n))$ means that $\lim_{n\to\infty} f(n)/g(n) = 0$

The statement in the question is equivalent to

$$\log(X_1 \cdots X_n)/n \to \alpha \quad \text{as } n \to \infty \text{ almost surely}.$$

Since $\log(X_1 \cdots X_n) = \log(X_1) + \cdots + \log(X_n)$ this statement is an immediate consequence of the strong law of large numbers applied to the sequence $(\log X_n)$.

(2) The idea is to consider random variables taking 2 values. Let X have law $(1-p)\delta_{1/2} + p\delta_{a'}$ where $p \in (0,1)$ and $a' > a$. We let (X_n) be a sequence of i.i.d. copies of X and we let $Y_n = X_1 \cdots X_n$. By (i), the sequence (Y_n) will have the desired property if

$$a = \mathbb{E}[X] = (1-p)/2 + pa' \quad \text{and} \quad 0 > \mathbb{E}[\log X] = (1-p)\log(1/2) + p\log(a').$$

The first equality can be rewritten in the form $p = (a-1/2)/(a'-1/2) \in (0,1)$. Hence it is enough to show that there is an $a' > a$ with

$$\frac{a'-a}{a'-1/2}\log(1/2) + \frac{a-1/2}{a'-1/2}\log(a') < 0.$$

The fact that such an $a' > a$ exists is clear since the right hand side of the display above goes to $\log(1/2) < 0$ as $a' \to \infty$.

$\square$

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. Let $(X_n)_{n\geq 1}$ be a sequence of real random variables. Show that there exists a sequence $c_n \to \infty$ such that X_n/c_n converges almost surely to 0.

Solution:

For fixed $n \geq 1$, since the cumulative distribution function of $|X_n|$ tends to 1 at $+\infty$, there exists $c_n > 0$ sufficiently large such that $\mathbb{P}\left(|X_n| > \frac{c_n}{n}\right) \leq 2^{-n}$. Let us show that (c_n) works.

By Borel-Cantelli, it is enough to show that for every $\epsilon > 0$, the series $\sum_{n\geq 1} \mathbb{P}\left(\left|\frac{X_n}{c_n}\right| > \epsilon\right)$ converges. To this end, for $n > 1/\epsilon$ (so that $\epsilon > 1/n$), write

$$\mathbb{P}\left(\left|\frac{X_n}{c_n}\right| > \epsilon\right) \leq \mathbb{P}\left(\left|\frac{X_n}{c_n}\right| > \frac{1}{n}\right) \leq \frac{1}{2^n}$$

by definition of c_n. This gives the desired result. $\square$

Exercise 7. Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence with the same law as X such that $\mathbb{E}[X^{2p}] < \infty$ for all integers $p \geq 1$. Also assume that $\mathbb{E}[X] = 0$.

(1) Show that for all integers $p \geq 1$ there exists a constant $C_p < \infty$ such that

$$\mathbb{E}((X_1 + \cdots + X_n)^{2p}) \leq C_p n^p .$$

(2) Deduce that $(X_1 + \cdots + X_n)/n^{1/2+\delta} \to 0$ as $n \to \infty$ almost surely for all $\delta > 0$.

Solution:

(1) We use the Binomial formula to write

$$\begin{aligned}\mathbb{E}((X_1 + \cdots + X_n)^{2p}) &= \mathbb{E}\left(\sum_{a_1+\cdots+a_n=2p} \binom{2p}{a_1,\ldots,a_n} \prod_{i=1}^{n} X_i^{a_i}\right) \\ &= \sum_{a_1+\cdots+a_n=2p} \binom{2p}{a_1,\ldots,a_n} \prod_{i=1}^{n} \mathbb{E}\left[X_i^{a_i}\right]\end{aligned} \tag{2}$$

where to go to the second line we used the linearity of the expectation and the independence of the (X_n). Let $C := 1 + \max\{\mathbb{E}[|X|^a] : 1 < a \leq 2p\}$. Since $\mathbb{E}[X] = 0$ all the terms in (2) with $a_i = 1$ for some i vanish, we get

$$\begin{aligned}\mathbb{E}\left((X_1+\cdots+X_n)^{2p}\right) &\leq \sum_{\substack{a_1+\cdots+a_n=2p\\ a_1,\ldots,a_n\neq 1}} \binom{2p}{a_1,\ldots,a_n} C^{\#\{i:\ a_i\neq 0\}}\\ &\leq C^{2p} \sum_{\substack{a_1+\cdots+a_n=2p\\ a_1,\ldots,a_n\neq 1}} \binom{2p}{a_1,\ldots,a_n}\end{aligned}$$

It only remains to bound the combinatorial factor. To this end, we split the sum up according to the indices i for which $a_i \geq 2$; note that the number of such indices is at most p since $a_1+\cdots+a_n = 2p$. We obtain

$$\begin{aligned}\sum_{\substack{a_1+\cdots+a_n=2p\\ a_1,\ldots,a_n\neq 1}} \binom{2p}{a_1,\ldots,a_n} &\leq \sum_{I\subset\{1,\ldots,n\}:\ \#I\leq p} \ \sum_{a_1+\cdots+a_{\#I}=2p} \binom{2p}{a_1,\ldots,a_{\#I}}\\ &= \sum_{m=1}^{p} \binom{n}{m} \sum_{a_1+\cdots+a_m=2p} \binom{2p}{a_1,\ldots,a_m}\\ &\leq n^p \sum_{m=1}^{p} \sum_{a_1+\cdots+a_m=2p} \binom{2p}{a_1,\ldots,a_m}\end{aligned}$$

and the claim follows since the final line is a product of n^p with a factor which only depends on p and not on n.

(2) Let us fix $\delta > 0$. By part (i) we can bound

$$\mathbb{E}\left(\sum_{n\geq 1}\left(\frac{X_1+\cdots+X_n}{n^{1/2+\delta}}\right)^{2p}\right) \leq \sum_{n\geq 1} C_p n^p \cdot n^{-p(1+2\delta)} = C_p \sum_{n\geq 1} n^{-2p\delta}$$

Let us choose p sufficiently large such that $2p\delta > 1$. Then the right hand side is finite and in particular

$$\sum_{n\geq 1}\left(\frac{X_1+\cdots+X_n}{n^{1/2+\delta}}\right)^{2p} < \infty \quad \text{almost surely}.$$

Consequently $(X_1+\cdots+X_n)/n^{1/2+\delta} \to 0$ as $n\to\infty$ almost surely as required.

□

Exercise 8. Find an integrable random variable X with $\mathbb{E}[X] = 0$ such that if $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with the same law as X then the series $\sum_{n\geq 1} X_n/n$ does not converge with positive probability.

You may assume that the converse of the Kolmogorov three series theorem is true.

Solution:

Consider $\alpha \in (0, \infty)$ and $p \in (0,1)$. We let X have law

$$\mathbb{P}_X = p \cdot \delta_{-1} + \frac{1-p}{c_\alpha} \sum_{k \geq 1} 2^{-k} k^{-1-\alpha} \delta_{2^k} \quad \text{where} \quad c_\alpha = \sum_{k \geq 1} 2^{-k} k^{-1-\alpha}$$

It is easy to see that we can pick $p \in (0,1)$ (depending on α) such that $\mathbb{E}[X] = 0$. We now show that the second condition of the three series criterion fails. Indeed, for $n > 1$ we have

$$\begin{aligned} -\mathbb{E}\left[X_n/n \mathbb{1}_{|X_n/n|<1}\right] &= \frac{-1}{n} \mathbb{E}\left[X \mathbb{1}_{|X|<n}\right] = \frac{1}{n} \mathbb{E}\left[X \mathbb{1}_{X \geq n}\right] \\ &= \frac{(1-p)/c_\alpha}{n} \sum_{k \geq 1} k^{-1-\alpha} \mathbb{1}_{2^k \geq n}. \end{aligned}$$

The inner sum can be lower bounded by $C_\alpha (\log n)^{-\alpha}$ for some constant $C_\alpha > 0$ depending on α. Therefore the second condition of the three series criterion is violated. □

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9. We color 10% of a sphere in blue and the rest in red. Show that it is possible to inscribe a cube in the sphere with all its vertices being red.

Solution:

To achieve this, we will use randomness! We randomly select a cube $A_1 A_2 \cdots A_8$ inscribed in the sphere (in a way that ensures that for all $1 \leq i \leq 8$, A_i follows a uniform distribution on the sphere). For $1 \leq i \leq 8$, we set $X_i = 1$ if A_i is red and $X_i = 0$ if A_i is blue.

For a fixed $1 \leq i \leq 8$, A_i follows a uniform distribution on the sphere. Therefore, $\mathbb{E}[X_i]$ is the probability that A_i is red, which is 0.9. By linearity of expectation,

$$\mathbb{E}\left[\sum_{i=1}^{8} X_i\right] = 8 \times 0.9 = 7.2.$$

Now, $\sum_{i=1}^{8} X_i$ is an integer random variable. Therefore, there exists a realization for which $\sum_{i=1}^{8} X_i = 8$ (otherwise, its expectation would be less than or equal to 7). □