Week 7: Different notions of convergence of random variables

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 6/11/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $\lambda > 0$ and let X be a real random variable such that $\mathbb{P}(X \geq a) = a^{-\lambda}$ for all $a \geq 1$.

(1) Show that X has a density and give its expression.

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with the same distribution as X. We define

$$T_n = \left(\prod_{i=1}^{n} X_i \right)^{1/n}.$$

(2) As $n \to \infty$, does T_n converge almost surely? Justify your answer.

(3) As $n \to \infty$, does T_n converge in probability? Justify your answer.

(4) Does $\mathbb{E}[T_n^2]$ converge as $n \to \infty$? Justify your answer.

(5) As $n \to \infty$, does T_n converge in L^1 ? Justify your answer.

2 Training exercises

Exercise 2. **(Scheffé Lemma)** Let $(X_n)_{n\geq 1}$ be *non-negative* real random variables that almost surely converge to X. We assume that $\mathbb{E}[X] < \infty$, and that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$. The goal of this exercise is to show that $X_n \to X$ in $\mathbb{L}^1$.

We define $Y_n = \min(X_n, X)$ and $Z_n = \max(X_n, X)$.

(1) Show that $\mathbb{E}[Y_n] \to \mathbb{E}[X]$ when $n \to \infty$.

(2) Show that $\mathbb{E}[Z_n] \to \mathbb{E}[X]$ when $n \to \infty$.

Hint. Write $Z_n = X + X_n - Y_n$.

(3) Conclude.

Hint. Write $|X_n - X| = Z_n - Y_n$.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables that converges in probability to X, and let $(Y_n)_{n\geq 1}$ be a sequence of real-valued random variables that converges in probability to Y. We want to show that (X_n, Y_n) converges in probability to (X, Y).

Exercise 4. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. real-valued random variables. Show that if $\mathbb{E}[|X1|] < \infty$, the sequence $(\max(X_1,\ldots,X_n)/n)_{n\geq 2}$ is uniformly integrable.

Exercise 5. We model the discretized evolution of a pollen particle between two absorbing plates as follows. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables with law given by $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. Let $k \geq 1$ be an integer. Set $S_0 = k$ and $S_n = k + X_1 + \cdots + X_n$ for $n \geq 1$. Define $T = \inf\{i \geq 1 : S_i = 0 \text{ or } S_i = 2k\}$ (with the convention $\inf \emptyset = \infty$).

(1) Show that $T < \infty$ almost surely.

(2) Set $Z_n = S_{\min(n,T)}$. Show that Z_n converges almost surely to a random variable, and determine its law. Does Z_n converge in probability? In $\mathbb{L}^1$?

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. (Coupon-collector problem) Let $(X_k, k \geq 1)$ be a sequence of independent random variables uniformly distributed over the set $\{1,2,\ldots,n\}$. Let

$$T_n = \inf\{m \geq 1 : \{X_1,\ldots,X_m\} = \{1,2,\ldots,n\}\}$$

the first time when all values have been observed.

(1) Set $\tau_k^n = \inf\{m \geq 1 : \mathrm{Card}(\{X_1,\ldots,X_m\}) = k\}$ for every $k \geq 1$. Show that the variables $(\tau_k^n - \tau_{k-1}^n)_{2\leq k\leq n}$ are independent and determine their respective distributions.

(2) Conclude that the convergence $\frac{T_n}{n\log n} \to 1$ holds in probability.

Hint. Show and use the Bienaymé-Tchebyshev inequality, which states that for every random variable Z and $x > 0$ we have $\mathbb{P}(|Z - \mathbb{E}[Z]| > x) \leq \frac{\mathrm{Var}(Z)}{x^2}$.

Exercise 7. Let (X_n) be a sequence of real random variables converging in probability to 0. Show that there exists a sequence $x_n \to 0$ such that $\mathbb{P}(|X_n| \geq x_n) \to 0$.

Exercise 8. Is the converse of Exercise 4 true? That is, if $(X_i)_{i\geq 1}$ are i.i.d. real-valued random variables, is it ture that if the sequence $(\max(X_1,\ldots,X_n)/n)_{n\geq 2}$ is uniformly integrable, then $\mathbb{E}[|X_1|] < \infty$?

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9. In the city of Knossos, there's a labyrinth with the following peculiarity: each room in the labyrinth has three corridors leading off it. King Minos places a Minotaur in the labyrinth, who performs the following routine over and over again: he walks down a corridor, enters a room and every other time takes the corridor on the right, and every other time the corridor on the left (. Show that the Minotaur will return to its initial point.