Week 7: Different notions of convergence of random variables

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 6/11/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $\lambda > 0$ and let X be a real random variable such that $\mathbb{P}(X \geq a) = a^{-\lambda}$ for all $a \geq 1$.

(1) Show that X has a density and give its expression.

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with the same distribution as X. We define

$$T_n = \left(\prod_{i=1}^{n} X_i \right)^{1/n}.$$

(2) As $n \to \infty$, does T_n converge almost surely? Justify your answer.

(3) As $n \to \infty$, does T_n converge in probability? Justify your answer.

(4) Does $\mathbb{E}[T_n^2]$ converge as $n \to \infty$? Justify your answer.

(5) As $n \to \infty$, does T_n converge in L^1 ? Justify your answer.

Solution:

(1) Observe that the cdf of X is given by $\mathbb{P}(X \leq a) = 1 - a^{-\lambda}$ for $a \geq 1$ and $\mathbb{P}(X \leq a) = 0$ for $a < 1$. The cdf is piece-wise C^1, so X has a density given by $-\mathbb{1}_{x\geq 1} \frac{\mathrm{d}}{\mathrm{d}x} x^{-\lambda} = \mathbb{1}_{x \geq 1} \frac{\lambda}{x^{\lambda+1}}$

(2) Yes, T_n converges almost surely. Observe that $\mathbb{P}(X \geq 1) = 1$, and that $\mathbb{P}(\ln(X) \geq a) = e^{-\lambda a}$ for every $a \geq 0$. Thus $\ln(X)$ follows an exponential law of parameter λ. In addition,

$$\ln(T_n) = \frac{1}{n} \sum_{i=1}^{n} \ln(X_i).$$

By the composition principle, the random variables $\ln(X_1), \ldots, \ln(X_n)$ are independent with same law distibuted as an exponential random variable of parameter λ. By the strong law of large numbers, $\ln(T_n)$ converges almost surely to $1/\lambda$. By continuity of the exponential function, it follows that T_n converges almost surely to $\exp(1/\lambda)$.

(3) Yes, T_n converges in probability to $\exp(1/\lambda)$: we saw in the lecture that almost sure convergence implies convergence in probability.

(4) Write

$$\mathbb{E}[T_n^2] = \mathbb{E}\left[\prod_{i=1}^n X_i^{2/n}\right] = \prod_{i=1}^n \mathbb{E}\left[X_i^{2/n}\right] = \mathbb{E}\left[X^{2/n}\right]^n,$$

where we have used the indepedence of $(X_i)_{1\leq i\leq n}$ for the second equality and the fact that these random variables all have the same law for the last equality. To compute $\mathbb{E}\left[X^{2/n}\right]$ using (1) and the transfer theorem:

$$\mathbb{E}\left[X^{2/n}\right] = \int_1^\infty x^{2/n} \cdot \frac{\lambda}{x^{\lambda+1}} \mathrm{d}x = \int_1^\infty \frac{\lambda}{x^{\lambda-2/n+1}} \mathrm{d}x$$

which is finite for n such that $\lambda - 2/n > 0$. Thus for n sufficiently large $\mathbb{E}\left[X^{2/n}\right] < \infty$ and

$$\mathbb{E}\left[X^{2/n}\right] = \frac{\lambda n}{\lambda n - 2} = 1 + \frac{2}{\lambda n - 2}.$$

Thus, using the Taylor expansion $\ln(1+x) = x + o(x)$ as $x \to 0$:

$$\mathbb{E}[T_n^2] = \left(1 + \frac{2}{\lambda n - 2}\right)^n = \exp\left(n \ln\left(1 + \frac{2}{\lambda n - 2}\right)\right) = \exp\left(n\left(\frac{2}{\lambda n - 2} + o\left(\frac{1}{n}\right)\right)\right) = \exp\left(\frac{2}{\lambda} + o(1)\right)$$

which converges to $\exp(2/\lambda)$ so the answer of the question is yes.

(5) The answer is yes.

Solution 1. We check that $(T_n)_{n\geq 1}$ converges in probability and is uniformly integrable. The convergence in probability has been established in (2) and uniform integrability comes from the fact that (T_n) is bounded in L^2 since $\mathbb{E}[T_n^2]$ converges as $n \to \infty$ (we saw in the lecture that a sequence of random variables bounded in L^p for $p > 1$ is uniformly integrable).

For Solutions 2 and 3, we first show that $\mathbb{E}[T_n] \to \exp(1/\lambda)$. As in question (4), we have

$$\mathbb{E}[T_n] = \mathbb{E}\left[\prod_{i=1}^n X_i^{1/n}\right] = \prod_{i=1}^n \mathbb{E}\left[X_i^{1/n}\right] = \mathbb{E}\left[X^{1/n}\right]^n,$$

and we similarly compute $\mathbb{E}\left[X^{1/n}\right]$:

$$\mathbb{E}\left[X^{1/n}\right] = \int_1^\infty x^{1/n} \cdot \frac{\lambda}{x^{\lambda+1}} \mathrm{d}x = \frac{\lambda n}{\lambda n - 1} = 1 + \frac{1}{\lambda n - 1}.$$

Thus, similarly to (4):

$$\mathbb{E}[T_n] = \left(1 + \frac{1}{\lambda n - 1}\right)^n = \exp\left(n \ln\left(1 + \frac{1}{\lambda n - 1}\right)\right) = \exp\left(n\left(\frac{1}{\lambda n - 1} + o\left(\frac{1}{n}\right)\right)\right) = \exp\left(\frac{1}{\lambda} + o(1)\right).$$

This entails

$$\mathbb{E}[T_n] \underset{n\to\infty}{\longrightarrow} \exp(1/\lambda),$$

Solution 2. We show that $\mathbb{E}[(T_n - \exp(1/\lambda)^2] \to 0$. Indeed, then by the Cauchy-Schwarz inequality $\mathbb{E}[|T_n - \exp(1/\lambda)|] \le \mathbb{E}[(T_n - \exp(1/\lambda))^2]^{1/2} \to 0$. To this end just write

$$\mathbb{E}[(T_n - \exp(1/\lambda)^2] = \mathbb{E}[T_n^2] - 2\exp(1/\lambda)\mathbb{E}[T_n] + \exp(2/\lambda) \underset{n\to\infty}{\longrightarrow} 0$$

since $\mathbb{E}[T_n^2] \to \exp(2/\lambda)$ and $\mathbb{E}[T_n] \to \exp(1/\lambda)$.

Solution 3. We have $T_n \ge 0$, $T_n \to \exp(1/\lambda)$ almost surely and $\mathbb{E}[T_n] \to \exp(1/\lambda)$. Then Scheffé's lemma (Exercise 2) implies that $T_n \to \exp(1/\lambda)$ in L^1.

□

2 Training exercises

Exercise 2. **(Scheffé Lemma)** Let $(X_n)_{n\ge 1}$ be *non-negative* real random variables that almost surely converge to X. We assume that $\mathbb{E}[X] < \infty$, and that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$. The goal of this exercise is to show that $X_n \to X$ in $\mathbb{L}^1$.

We define $Y_n = \min(X_n, X)$ and $Z_n = \max(X_n, X)$.

(1) Show that $\mathbb{E}[Y_n] \to \mathbb{E}[X]$ when $n \to \infty$.

(2) Show that $\mathbb{E}[Z_n] \to \mathbb{E}[X]$ when $n \to \infty$.
Hint. Write $Z_n = X + X_n - Y_n$.

(3) Conclude.
Hint. Write $|X_n - X| = Z_n - Y_n$.

Solution:

(1) Since $X_n \to X$ almost surely, we have that Y_n converges almost surely to X. In addition $|Y_n| \le X$. Thus $\mathbb{E}[Y_n] \to \mathbb{E}[X]$ by the dominated convergence theorem.

(2) We have $\mathbb{E}[Z_n] = \mathbb{E}[X] + \mathbb{E}[X_n] - \mathbb{E}[Y_n] \to \mathbb{E}[X] + \mathbb{E}[X] - \mathbb{E}[X] = \mathbb{E}[X]$.

(3) We have $\mathbb{E}[|X_n - X|] = \mathbb{E}[Z_n] - \mathbb{E}[Y_n] \to \mathbb{E}[X] - \mathbb{E}[X] = 0$. This shows that $X_n \to X$ in L^1.

Remark. Convergence in L^1 implies convergence of expectations: indeed, if $X_n \to X$ in L^1, then $|\mathbb{E}[X_n] - \mathbb{E}[X]| \le \mathbb{E}[|X_n - X|] \to 0$. However, we can have convergence of expectations without convergence in L^1: for instance, taking the example from the lecture $X_n(\omega) = 2^n \mathbb{1}_{\omega\in[0,1/2^n]}$ on probability space $([0,1], \mathcal{B}[0,1], \mathsf{Leb}_{[0,1]})$, we have convergence of expectations since $\mathbb{E}(X_n) = 1$ is constant for all n, but X_n does not convergence in L^1. Indeed, argue by contradiction: if $X_n \to X$ in L^1, this would imply that $X_n \to X$ in probability. But $X_n \to 0$ in probability. Thus $X = 0$ almost surely, so $X_n \to 0$ in L^1, which implies $\mathbb{E}[X_n] \to 0$, which is a contradiction. $\square$

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables that converges in probability to X, and let $(Y_n)_{n\geq 1}$ be a sequence of real-valued random variables that converges in probability to Y. We want to show that (X_n, Y_n) converges in probability to (X, Y).

Solution:

First solution. For any $\varepsilon > 0$, we observe that if

$$\|(X_n, Y_n) - (X, Y)\| \geq 2\varepsilon.$$

then either $|X_n - X| \geq \varepsilon$ or $|Y_n - Y| \geq \varepsilon$. Indeed, by contradiction, if $|X_n - X| < \varepsilon$ and $|Y_n - Y| < \varepsilon$, then

$$\|(X_n, Y_n) - (X, Y)\| \leq \sqrt{\varepsilon^2 + \varepsilon^2} = \sqrt{2}\varepsilon < 2\varepsilon.$$

Thus, for any $\varepsilon > 0$,

$$\mathbb{P}(\|(X_n, Y_n) - (X, Y)\| \geq 2\varepsilon) \leq \mathbb{P}(|X_n - X| \geq \varepsilon) + \mathbb{P}(|Y_n - Y| \geq \varepsilon) \quad \underset{n\to\infty}{\longrightarrow} \quad 0,$$

which implies the result.

Second solution. We use the sub-sub-sequence lemma: given a subsequence φ, we find ψ such that $(X_{\varphi\circ\psi(n)}, Y_{\varphi\circ\psi(n)})$ converges almost surely to (X, Y).

First, since $X_n \to X$ in probability, we can find a σ_1 suh that $X_{\varphi\circ\sigma_1(n)}$ converges almost surely to X. But $Y_{\varphi\circ\sigma_1(n)}$ converges in probability to Y, so we can find σ_2 such that $Y_{\varphi\circ\sigma_1\circ\sigma_2(n)}$ converges almost surely to Y. But then $X_{\varphi\circ\sigma_1\circ\sigma_2(n)}$ also converges almost surely to X, so we can take $\psi = \sigma_1 \circ \sigma_2$. $\square$

Exercise 4. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. real-valued random variables. Show that if $\mathbb{E}[|X1|] < \infty$, the sequence $(\max(X_1, \ldots, X_n)/n)_{n\geq 2}$ is uniformly integrable.

Solution:

First solution. We show uniform integrability via its characterization of boundedness in L^1 and $\epsilon - \delta$ condition. For boundedness in L^1, we have for all $n \geq 2$,

$$\mathbb{E}\left[\left|\frac{\max(X_1, \ldots, X_n)}{n}\right|\right] \leq \mathbb{E}\left[\frac{\sum_{k=1}^n |X_k|}{n}\right] = \mathbb{E}[|X_1|] < \infty.$$

For $\epsilon - \delta$ condition, first notice that $\{X_1\}$ is a finite family of integrable random variables, hence this family is uniformly integrable. So $\{X_1\}$ satisfies the $\epsilon - \delta$, which says $\forall\epsilon > 0, \exists\delta > 0, \forall A \in \mathcal{A}$ with

$\mathbb{P}(A) < \delta$, we have $\mathbb{E}(|X_1|\mathbb{1}_A) < \epsilon$. Then we have for all $n \geq 2$,

$$\mathbb{E}\left[\left|\frac{\max(X_1,\ldots,X_n)}{n}\mathbb{1}_A\right|\right] \leq \mathbb{E}\left[\frac{\sum_{k=1}^n |X_k|}{n}\mathbb{1}_A\right] = \mathbb{E}(|X_1|\mathbb{1}_A) < \epsilon.$$

We conclude that family the $\{\frac{\max(X_1,\ldots,X_n)}{n}\}_{n\geq 2}$ satisfies the $\epsilon-\delta$ condition.

Second solution. Write

$$\left|\frac{\max(X_1,\ldots,X_n)}{n}\right| \leq \frac{\sum_{k=1}^n |X_k|}{n}.$$

We claim that the sequence $\frac{\sum_{k=1}^n |X_k|}{n}$ converges in L^1 to $\mathbb{E}[|X_1|]$. We show this by using Scheffé's Lemma (exercise 2):

– the random variables are nonnegative and the convergence holds a.s. by the strong law of large numbers since X_1 is integrable.

– $\mathbb{E}\left[\frac{\sum_{k=1}^n |X_k|}{n}\right] = \mathbb{E}[|X_1|]$ which converges to $\mathbb{E}[|X_1|]$ (it is a constant sequence).

But if $0 \leq |Y_n| \leq Z_n$ and if Z_n converges in L^1 then (Y_n) is uniformly integrable. Indeed, we have seen in the lecture the fact that (Z_n) converges in L^1 implies that (Z_n) is uniformly integrable, and since we have for every $K > 0$

$$\mathbb{E}\left[|Y_n|\mathbb{1}_{|Y_n|\geq K}\right] \leq \mathbb{E}\left[Z_n\mathbb{1}_{Z_n\geq K}\right],$$

this shows that (Y_n) is uniformly integrable.

The desired result follows by appling this with $Y_n = \frac{\max(X_1,\ldots,X_n)}{n}$ and $Z_n = \frac{\sum_{k=1}^n |X_k|}{n}$.

Remark. We will later see in class that the convergence in the strong law of large numbers also always holds in L^1. □

Exercise 5. We model the discretized evolution of a pollen particle between two absorbing plates as follows. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables with law given by $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. Let $k \geq 1$ be an integer. Set $S_0 = k$ and $S_n = k + X_1 + \cdots + X_n$ for $n \geq 1$. Define $T = \inf\{i \geq 1 : S_i = 0 \text{ or } S_i = 2k\}$ (with the convention $\inf\emptyset = \infty$).

(1) Show that $T < \infty$ almost surely.

(2) Set $Z_n = S_{\min(n,T)}$. Show that Z_n converges almost surely to a random variable, and determine its law. Does Z_n converge in probability? In $\mathbb{L}^1$?

Solution:

(1) If there are $2k$ consecutive upward sets, then $T < \infty$. An application of Borel–Cantelli shows that in the sequence $(X_i)_{i\geq 1}$ the number 1 appears $2k$ times consecutively infinitely often almost surely. Indeed, for $i \geq 1$ set

$$A_i = \{X_{2ki} = 1, X_{2ki+1} = 1, \ldots, X_{2ki+2k-1} = 1\}.$$

These events are independent by the coalition principle, and $\mathbb{P}(A_i) = \frac{1}{2^{2k}}$. Thus $\sum_{i\geq 1}\mathbb{P}(A_i) = \infty$, so by the second Borel–Cantelli lemma almost surely the events A_i happen infinitely many times. Hence $\mathbb{P}(T < \infty) = 1$.

(2) Since $T < \infty$ almost surely, Z_n converges almost surely to S_T. Hence Z_n converges also in probability to S_T (recall that almost sure convergence implies convergence in probability). Since $|Z_n| \leq 2k$, the convergence also holds in $\mathbb{L}^1$.

By symmetry,

$$\mathbb{P}(S_T = 0) = \mathbb{P}(S_T = 2k) = \frac{1}{2}.$$

Let us write this argument in a more formal way. Denote by Φ the map which transforms a walk on $\mathbb{Z}$ which makes ± 1 jumps by transforming -1 jumps into $+1$ jumps. Set $(S'_n)_{n\geq 0} = \Phi((S_n)_{n\geq 0})$ and $T' = \inf\{i \geq 1 : S'_i = 0 \text{ or } S'_i = 2k\}$. Hence, by construction, $S_T = 0$ if and only if $S'_{T'} = 2k$. In addition, $(S_n)_{n\geq 0}$ and $(S'_n)_{n\geq 0}$ have the same law. Hence $\mathbb{P}(S_T = 0) = \mathbb{P}\left(S'_{T'} = 2k\right) = \mathbb{P}(S_T = 2k)$. Since $\mathbb{P}(S_T = 0) + \mathbb{P}(S_T = 2k) = 1$, the desired result follows.

□

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. (Coupon-collector problem) Let $(X_k, k \geq 1)$ be a sequence of independent random variables uniformly distributed over the set $\{1, 2, \ldots, n\}$. Let

$$T_n = \inf\{m \geq 1 : \{X_1, \ldots, X_m\} = \{1, 2, \ldots, n\}\}$$

the first time when all values have been observed.

(1) Set $\tau_k^n = \inf\{m \geq 1 : \mathrm{Card}(\{X_1, \ldots, X_m\}) = k\}$ for every $k \geq 1$. Show that the variables $(\tau_k^n - \tau_{k-1}^n)_{2\leq k\leq n}$ are independent and determine their respective distributions.

(2) Conclude that the convergence $\frac{T_n}{n \log n} \to 1$ holds in probability.

Hint. Show and use the Bienaymé-Tchebyshev inequality, which states that for every random variable Z and $x > 0$ we have $\mathbb{P}(|Z - \mathbb{E}[Z]| > x) \leq \frac{\mathrm{Var}(Z)}{x^2}$.

Solution:

(1) The quantity $\tau_k^n - \tau_{k-1}^n$ represents the time taken to obtain a new element once you have obtained $k-1$ elements. Intuitively, these variables are independent and follow a geometric distribution with a parameter of $\frac{n-k+1}{n}$ (there are $n-(k-1)$ elements remaining).

To formally prove this, we can proceed as follows. We have $\tau_1^n = 1$. Let $(t_2, \ldots, t_n) \in (\mathbb{N}^*)^{n-1}$. We want to calculate $\mathbb{P}(\tau_2^n - \tau_1^n = t_2, \ldots, \tau_n^n - \tau_{n-1}^n = t_n)$.

By setting $t_1 = 1$ and by letting $\mathcal{S}_n$ be the set of permutations of $1, 2, \ldots, n$, we have:

$$\begin{aligned}
&\mathbb{P}(\tau_2^n - \tau_1^n = t_2, \ldots, \tau_n^n - \tau_{n-1}^n = t_n) \\
&= \sum_{\sigma \in \mathcal{S}_n} \mathbb{P}\left(\bigcap_{k=1}^{n-1} \Big\{ X_{t_1+\ldots+t_k} = \sigma(k), X_{t_1+\ldots+t_k+1} \in \{\sigma(1), \ldots, \sigma(k)\}, \ldots, \right. \\
&\qquad\qquad \left. X_{t_1+\ldots+t_k+t_{k+1}-1} \in \{\sigma(1), \ldots, \sigma(k)\} \Big\} \cap \{X_{t_1+\ldots+t_n} = \sigma(n)\} \right) \\
&= \sum_{\sigma \in \mathcal{S}_n} \frac{1}{n^n} \prod_{k=2}^{n} \left(\frac{k-1}{n} \right)^{t_k - 1} \\
&= \frac{n!}{n^n} \prod_{k=2}^{n} \left(\frac{k-1}{n} \right)^{t_k - 1} \\
&= \prod_{k=2}^{n} \left(\frac{n+1-k}{n} \right) \left(\frac{k-1}{n} \right)^{t_k - 1}.
\end{aligned}$$

Therefore, the random variables $(\tau_k^n - \tau_{k-1}^n)_{2 \leq k \leq n}$ are independent and have the following respective distribution:

$$\mathbb{P}\left(\tau_k^n - \tau_{k-1}^n = i \right) = \left(\frac{n+1-k}{n} \right) \left(\frac{k-1}{n} \right)^{i-1} = \left(\frac{n-k+1}{n} \right) \left(1 - \frac{n-k+1}{n} \right)^{i-1} \qquad \text{for } i \geq 1.$$

This distribution is indeed the distribution of G_k where G_k follows a geometric distribution with a parameter of $\frac{n-k+1}{n}$.

(2) To prove the Bienaymé-Tchebyshev inequality, write using Markov's inequality:

$$\mathbb{P}(|Z - \mathbb{E}[Z]| > x) = \mathbb{P}((Z - \mathbb{E}[Z])^2 > x^2) \leq \frac{\mathbb{E}[(Z - \mathbb{E}[Z])^2]}{x^2} = \frac{\mathsf{Var}(Z)}{x^2}.$$

Now, we have $T_n = 1 + \sum_{k=2}^{n} (\tau_k^n - \tau_{k-1}^n)$, and therefore:

$$\mathbb{E}(T_n) = 1 + \sum_{k=2}^{n} \frac{n}{n+1-k} = 1 + nH_{n-1},$$

where H_n is the harmonic series, and using the fact that the variance of a geometric random variable with parameter p is $(1-p)/p^2$, we get

$$\mathrm{Var}(T_n) = \sum_{k=2}^{n} \mathrm{Var}(\tau_k^n - \tau_{k-1}^n) = \sum_{k=2}^{n} \frac{(k-1)/n}{((n+1-k)/n)^2} = n \sum_{k=1}^{n-1} \frac{n-k}{k^2} \leq Cn^2$$

with $C = \sum_{k\geq 1} \frac{1}{k^2}$. So for any $\varepsilon > 0$:

$$\mathbb{P}\left(|T_n - \mathbb{E}(T_n)| \geq \varepsilon n \log(n)\right) \leq \frac{\mathrm{Var}(T_n)}{\varepsilon^2 n^2 \log(n)^2} \leq \frac{C}{\varepsilon^2 \log(n)^2}.$$

Therefore, $(T_n - \mathbb{E}(T_n))/(n\log(n)) \to 0$ in probability. Now, $\mathbb{E}(T_n) \sim n\log(n)$ as n approaches infinity. So, for any $\varepsilon > 0$, $\{|T_n - n\log(n)| \geq 2\varepsilon n \log(n)\} \subset \{|T_n - \mathbb{E}(T_n)| \geq \varepsilon n \log(n)\}$ for sufficiently large n. This yields the result.

□

Exercise 7. Let (X_n) be a sequence of real random variables converging in probability to 0. Show that there exists a sequence $x_n \to 0$ such that $\mathbb{P}(|X_n| \geq x_n) \to 0$.

Solution:

Let i_k be such that $\mathbb{P}(|X_n| \geq \frac{1}{k}) \leq \frac{1}{k}$ for $n \geq i_k$. Without loss of generality, we may assume that (i_k) is increasing. For $n \geq 1$, we then set

$$x_n = \frac{1}{i_k} \qquad \text{with } i_k \leq n < i_{k+1}.$$

Then clearly $x_n \to 0$ since $i_k \to \infty$, and for $n \geq 1$, if k_n is such that $i_{k_n} \leq n < i_{k_n+1}$, we have

$$\mathbb{P}(|X_n| \geq x_n) = \mathbb{P}\left(|X_n| \geq \frac{1}{i_{k_n}}\right) \leq \frac{1}{i_{k_n}} \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

□

Exercise 8. Is the converse of Exercise 4 true? That is, if $(X_i)_{i\geq 1}$ are i.i.d. real-valued random variables, is it ture that if the sequence $(\max(X_1,\ldots,X_n)/n)_{n\geq 2}$ is uniformly integrable, then $\mathbb{E}[|X_1|] < \infty$?

Solution:

Yes, if the random variables are non-negative. In such cases, $0 \leq X_1/2 \leq \max(X_1, X_2)/2$, and as a result, $\mathbb{E}[X_1] < \infty$.

In general, no! For example, if we take X_1 to be a random variable with density given by $\frac{1}{|x|^2}\mathbb{1}_{x\leq -1}$, we can observe that $\max(X_1,\ldots,X_n)$ has density $\frac{n}{|x|^{n+1}}\mathbb{1}_{x\leq -1}$, and as a result,

$$\mathbb{E}\left[\left|\frac{\max(X_1,\ldots,X_n)}{n}\right|^{3/2}\right] = \frac{1}{n^{3/2}}\int_1^\infty \frac{nx^{3/2}}{x^{n+1}}\mathrm{d}x = \frac{1}{n^{3/2}}\frac{2n}{2n-3}.$$

The sequence $(\frac{\max(X_1,\ldots,X_n)}{n})n \geq 2$ being bounded in $L^{3/2}$, it is uniformly integrable, even though X_1 is not integrable.

□

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9. In the city of Knossos, there's a labyrinth with the following peculiarity: each room in the labyrinth has three corridors leading off it. King Minos places a Minotaur in the labyrinth, who performs the following routine over and over again: he walks down a corridor, enters a room and every other time takes the corridor on the right, and every other time the corridor on the left (. Show that the Minotaur will return to its initial point.

Solution:
The idea is to consider the set of all configurations of the form (s_1, s_2, d), where s_1 indicates the room the minotaur comes from, s_2 the room he's in, and d is the direction he must take (right or left).

Observe that the set of all configurations is a finite set, so the Minotaur makes a cycle in this set. In addition, we can reconstruct the past of the minotaur's journey from any element in this cycle, so all its past journey belongs to this cycle. In particular, the Minotaur returns to the initial room. □