Week 8: Conditional expectation

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 13/11/2023 17:00 (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/`

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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All random variables are assumed to be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. The goal of this exercise is to compute some conditional expectations (please justify your computions).

(1) Let $(X_i)_{1\leq i\leq n}$ be i.i.d. integrable random variables and set $S_n = X_1 + X_2 + \cdots + X_n$.

   (a) Compute $\mathbb{E}[S_n \mid X_1]$.

   (b) Compute $\mathbb{E}[X_1 \mid S_n]$.

(2) Let X and Y be two independent Bernoulli distributed random variables with parameter $p \in [0,1]$. Set $Z = \mathbb{1}_{X+Y=0}$.

   (a) Compute $\mathbb{E}[X \mid Z]$ and $\mathbb{E}[Y \mid Z]$.

   (b) Are these two random variables independent? Justify your answer.

2 Training exercises

Exercise 2. Let X be an integrable random variable and $\mathcal{A} \subset \mathcal{F}$ a σ-field. Let Y be an integrable $\mathcal{A}$-measurable random variable.

(1) Show that $Y = \mathbb{E}[X \mid \mathcal{A}]$ if and only if for every $A \in \mathcal{A}$, $\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$.

(2) Let $\mathcal{C} \subset \mathcal{A}$ be a generating π-system of $\mathcal{A}$ containing Ω. Using the Dynkin Lemma, show that $Y = \mathbb{E}[X \mid \mathcal{A}]$ if and only if for every $A \in \mathcal{C}$, $\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$.

Exercise 3. We consider a population in which there is a large number n of households. We model the size of the households by i.i.d. random variables $(X_i)_{1\leq i\leq n}$ on $\mathbb{N}^*$, with mean $m = \mathbb{E}[X_1] = \sum_{k\geq 1} kp_k < \infty$ where $p_k = \mathbb{P}(X_1 = k)$. Let T_n be the size of the household of an individual chosen uniformly at random in the population.

(1) Justify that for every integer $k \geq 1$ we have

$$\mathbb{P}(T_n = k \mid X_1, \ldots, X_n) = \frac{k \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i = k\}}}{\frac{1}{n} \sum_{i=1}^{n} X_i}.$$

(2) Show that for every integer $k \geq 1$, $\mathbb{P}(T_n = k)$ converges to $\frac{k}{m} p_k$ as $n \to \infty$.

Exercise 4. Let X, Y be two random variables with values in respectively E, F. Let $\mathcal{A}$ be a sub σ-field of $\mathcal{F}$. Assume that X is independent from $\mathcal{A}$ and that Y is $\mathcal{A}$-measurable. Show that for any measurable function $g : E \times F \to \mathbb{R}^+$, we have

$$\mathbb{E}[g(X,Y) \mid \mathcal{A}] = h(Y) \qquad \text{a.s.,} \qquad \text{where} \qquad h(y) = \mathbb{E}[g(X,y)].$$

Remark. In particular, this applies to $\mathbb{E}[g(X,Y) \mid Y]$ when X and Y are independent. Intuitively speaking, in this case, the conditional expectation is done by first computing the expectation “with respect to X” (i.e. by integrating with respect to the law of X) by considering Y as being “fixed”, and then computing the expectation of the result “with respect to Y” (i.e. by integrating with respect to the law of Y).

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let $\mathcal{A} \subset \mathcal{F}$ be a σ-field, and X a nonnegative random variable. Show that $\{\mathbb{E}[X \mid \mathcal{A}] > 0\}$ is the smallest $\mathcal{A}$-measurable set (up to negligeable sets, that is events with 0 probability) containing $\{X > 0\}$.

Exercise 6. Let $(X_i)_{i \geq 1}$ be a sequence of nonnegative real random variables and $(\mathcal{F}_i)_{i \geq 1}$ a sequence of σ-fields included in $\mathcal{F}$. Assume that $\mathbb{E}[X_i \mid \mathcal{F}_i]$ converges in probability to 0.

(1) Show that $(X_i)_{i \geq 1}$ converges in probability to 0.
(2) Show that the converse is false.

Exercise 7. Let $\mathcal{A} \subset \mathcal{F}$ be a σ-field. We say that two random variables X and Y are independent conditionally given $\mathcal{A}$ if for every nonnegative measurable functions $f, g : \mathbb{R} \to \mathbb{R}_+$ we have

$$\mathbb{E}[f(X)g(Y) \mid \mathcal{A}] = \mathbb{E}[f(X) \mid \mathcal{A}]\mathbb{E}[g(Y) \mid \mathcal{A}]. \tag{1}$$

(1) What does this mean if $\mathcal{A} = \{\emptyset, \Omega\}$? If $\mathcal{A} = \mathcal{F}$?
(2) Show that the previous definition (1) is equivalent to (a) and that (1) is equivalent to (b):
  (a) for every nonnegative $\mathcal{A}$-measurable random variable Z, for every nonnegative measurable functions $f, g : \mathbb{R} \to \mathbb{R}_+$, we have $\mathbb{E}[f(X)g(Y)Z] = \mathbb{E}[f(X)Z\mathbb{E}[g(Y) \mid \mathcal{A}]]$,
  (b) for every nonnegative measurable function $g : \mathbb{R} \to \mathbb{R}_+$ we have $\mathbb{E}[g(Y) \mid \sigma(\mathcal{A}, \sigma(X))] = \mathbb{E}[g(Y) \mid \mathcal{A}]$.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 8. Imagine there are a 100 people in line to board a plane that seats 100. The first person in line, Alice, realizes she lost her boarding pass, so when she boards she decides to take a random seat instead.

Every person that boards the plane after her will either take their ”proper” seat, or if that seat is taken, a random seat instead.

What is the probability that the last person that boards will end up in their proper seat?