Week 9: (sub/super)martingales and their a.s. convergence

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 20/11/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. (Pólya's Urn) At time 0, an urn contains 1 black ball and 1 white ball. At each time $n \geq 1$ a ball is chosen uniformly at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. We let $\mathcal{F}_n = \sigma(B_1, \ldots, B_n)$ for $n \geq 1$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(1) For every $n \geq 1$ prove that B_n is uniformly distributed on $\{0, 1, \ldots, n\}$.

Hint. Argue by induction.

(2) Let $M_n = (B_n + 1)/(n+2)$ be the proportion of black balls in the urn just after time n. Prove that (M_n) is a martingale with respect to $(\mathcal{F}_n)$ and show that $M_n \to U$ as $n \to \infty$ a.s. for some random variable U.

(3) Show that U follows the uniform distribution on $(0, 1)$.

Hint. For a continuous function $f : [0, 1] \to \mathbb{R}$, consider $f(M_n)$ and use Exercise 1 from Exercise Sheet 4.

(4) Fix $0 < \theta < 1$ and define for $n \geq 0$

$$N_n = \frac{(n+1)!}{B_n!(n-B_n)!} \theta^{B_n} (1-\theta)^{n-B_n}$$

Show that $(N_n)_{n \geq 0}$ is a martingale for the filtration $(\mathcal{F}_n)_{n \geq 0}$.

2 Training exercises

Exercise 2. Let (M_n) be a submartingale such that

$$\sup_{n\geq 1} \mathbb{E}[M_n^+] < \infty$$

where $M_n^+ = \max(M_n, 0)$. Show that $(M_n)_{n\geq 0}$ converges almost surely.

Exercise 3. Let $(X_i)_{i\geq 1}$ be i.i.d. random variables with values in $\{-1, 1\}$ where we write $\mathbb{P}(X_i = 1) = p$ and assume that $p \in (0, 1/2)$. Moreover, define $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. For $n \geq 0$ we set

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}.$$

For $n \geq 1$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(1) Show that (M_n) is an L^1 bounded martingale with respect to the filtration $(\mathcal{F}_n)$ where.

(2) Show that M_n converges almost surely to 0 as $n \to \infty$, but does not converge in L^1.

Exercise 4.

(1) Find an example of a martingale which is not bounded in L^1.

(2) Find an example of a martingale which converges almost surely but which is not bounded in L^1.

(3) Find an example of a martingale which converges almost surely to ∞.

Hint. Search for martingales of the form $M_n = X_1 + \cdots + X_n$.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let $(Y_n)_{n\geq 0}$ be a sequence of non-negative i.i.d. random variables with $\mathbb{E}(Y_1) = 1$ and $\mathbb{P}(Y_1 = 1) < 1$. For $n \geq 1$ we let $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$, and we set $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(1) Show that $X_n = \prod_{i=1}^n Y_i$ defines a martingale with respect to $(\mathcal{F}_n)$.

(2) Show that $X_n \to 0$ as $n \to \infty$ a.s.

Hint. You may use the strict Jensen inequality: if $f : \mathbb{R} \to \mathbb{R}$ is strictly convex and X is a non-constant random variable such that X and $f(X)$ are integrable, then $f(\mathbb{E}[X]) < \mathbb{E}[f(X)]$.

Exercise 6. (Bellman's Optimality Principle) Your winnings per unit stake on game n are ϵ_n, where the ϵ_n are i.i.d. random variables with

$$\mathbb{P}(\epsilon_n = +1) = p, \quad \mathbb{P}(\epsilon_n = -1) = q, \quad \text{where } 1/2 < p = 1 - q < 1.$$

Your stake C_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Your objective is to maximize the expected *interest rate* $\mathbb{E}[\ln(Z_N/Z_0)]$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0, is a given constant. Let $\mathcal{F}_n = \sigma(\epsilon_1, \dots, \epsilon_n)$ be your *history* up to time n. We assume that $\ln(Z_n)$ is integrable for all $n \geq 0$.

(1) Show that if C is any (predictable) strategy, meaning that for all $n \geq 1$, C_n is $\mathcal{F}_{n-1}$ measurable, then $\ln(Z_n) - n\alpha$ is a supermartingale, where α denotes the *entropy*

$$\alpha = p \ln p + q \ln q + \ln 2.$$

Conclude that $\mathbb{E}[\ln(Z_N/Z_0)] \leq N\alpha$.

(2) Show that for a certain strategy, $\ln(Z_n) - n\alpha$ is a martingale. What is the best strategy?

Exercise 7. Find a sequence $(M_n)_{n\geq 0}$ of integrable random variables such that $\mathbb{E}[M_{n+1}|M_n] = M_n$ but such that $(M_n)_{n\geq 0}$ is not a martingale with respect to its canonical filtration.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 8. A mathematician, an economist and a trader are chatting in a bar. The economist says:

"The euro value of a CHF over time is a martingale! Otherwise, it would be possible to make money on average, buying and selling CHF at the right time!"

The mathematician replies: "But if that's true, according to conditional's Jensen's inequality, the CHF value of a euro is a sub-martingale!"

The trader says nothing, thinks for a few seconds, then runs off to buy euros.

What do you think?