Week 9: (sub/super)martingales and their a.s. convergence

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 20/11/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

***

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. (Pólya's Urn) At time 0, an urn contains 1 black ball and 1 white ball. At each time $n \geq 1$ a ball is chosen uniformly at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. We let $\mathcal{F}_n = \sigma(B_1, \dots, B_n)$ for $n \geq 1$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(1) For every $n \geq 1$ prove that B_n is uniformly distributed on $\{0, 1, \dots, n\}$.

Hint. Argue by induction.

(2) Let $M_n = (B_n + 1)/(n+2)$ be the proportion of black balls in the urn just after time n. Prove that (M_n) is a martingale with respect to $(\mathcal{F}_n)$ and show that $M_n \to U$ as $n \to \infty$ a.s. for some random variable U.

(3) Show that U follows the uniform distribution on $(0, 1)$.

Hint. For a continuous function $f : [0, 1] \to \mathbb{R}$, consider $f(M_n)$ and use Exercise 1 from Exercise Sheet 4.

(4) Fix $0 < \theta < 1$ and define for $n \geq 0$

$$N_n = \frac{(n+1)!}{B_n!(n-B_n)!} \theta^{B_n} (1-\theta)^{n-B_n}$$

Show that $(N_n)_{n \geq 0}$ is a martingale for the filtration $(\mathcal{F}_n)_{n \geq 0}$.

Solution:

(1) We prove the claim by induction; when $n = 0$ the claim is obvious. Let us now consider the induction step. Then by the problem description, for $b \in \{0, \dots, n+1\}$, almost surely we have

$$\mathbb{P}(B_{n+1} = b \mid \mathcal{F}_n) = \frac{B_n + 1}{n+2} \mathbb{1}_{b = B_n + 1} + \frac{n+1-B_n}{n+2} \mathbb{1}_{b = B_n}. \qquad (1)$$

(recall that by definition $\mathbb{P}(A|\mathcal{F}_n) = \mathbb{E}[\mathbb{1}_A|\mathcal{F}_n]$ for any event A. Thus by taking expectations on both sides and using the fact that B_n is uniform on $\{0,\dots,n\}$ we get

$$\mathbb{P}(B_{n+1} = b) = \frac{b}{n+2} \cdot \frac{1}{n+1} + \frac{n+1-b}{n+2} \cdot \frac{1}{n+1} = \frac{1}{n+2}$$

as required.

(2) Clearly $M_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ since it is $\mathcal{F}_n$-measurable with values in $[0,1]$. By (1) we get

$$\begin{aligned}\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= \sum_{b=0}^{n+2} \frac{b+1}{n+3} \mathbb{P}(B_{n+1} = b \mid \mathcal{F}_n) \\ &= \frac{B_n+2}{n+3} \cdot \frac{B_n+1}{n+2} + \frac{B_n+1}{n+3} \cdot \frac{n+1-B_n}{n+2} = M_n \quad \text{a.s.}\end{aligned}$$

Since (M_n) is bounded in L^1 the martingale convergence theorem implies that the limit $U = \lim_{n\to\infty} M_n$ almost surely exists.

(3) Let $f : [0,1] \to \mathbb{R}$ be continuous. We have

$$\mathbb{E}[f(M_n)] = \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+2}\right) \quad \underset{n\to\infty}{\longrightarrow} \quad \int_0^1 f(t)\mathrm{d}t, \tag{2}$$

by the Riemann theorem. This can be proved by hand: since f is continuous on $[0,1]$ it is also uniformly continuous, so for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x-y| \leq \delta$ implies $|f(x)-f(y)| \leq \varepsilon$. Then, for n such that $1/n < \delta$ we have $|\frac{i}{n+2} - \frac{i}{n+1}| \leq \frac{1}{n}$, so

$$\left| \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+2}\right) - \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) \right| \leq \varepsilon$$

and also

$$\begin{aligned}\left| \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) - \int_0^1 f(t)\mathrm{d}t \right| &= \left| \sum_{i=1}^{n+1} \int_{\frac{i-1}{n+1}}^{\frac{i}{n+1}} \left(f\left(\frac{i}{n+1}\right) - f(t)\mathrm{d}t \right) \right| \\ &\leq \sum_{i=1}^{n+1} \int_{\frac{i-1}{n+1}}^{\frac{i}{n+1}} \left| f\left(\frac{i}{n+1}\right) - f(t)\mathrm{d}t \right| \\ &\leq \varepsilon.\end{aligned}$$

which implies (2).

But $M_n \to U$ almost surely, so by continuity of f we also have $f(M_n) \to f(U)$ almost surely. Since f is continuous on $[0,1]$ is it bounded, so by dominated convergence we have $\mathbb{E}[f(M_n)] \to \mathbb{E}[f(U)]$.

By (2) we conclude with the transfer theorem that

$$\int_{\mathbb{R}} f(x)\mathbb{P}_U(\mathrm{d}x) = \mathbb{E}[f(U)] = \int_0^1 f(x)\mathrm{d}x.$$

By Exercise 1(3) of Exercise Sheet 4 we conclude that U follows the uniform distribution on $[0,1]$.

(4) For $n \geq 0$ the random variable N_n^θ is $\mathcal{F}_n$ measurable and non-negative. By (1) we deduce that

$$\begin{aligned}
\mathbb{E}(N_{n+1} \mid \mathcal{F}_n) &= \sum_{b=0}^{n+2} \frac{(n+2)!}{b!(n+1-b)!}\theta^b(1-\theta)^{n+1-b}\,\mathbb{P}(B_{n+1}=b \mid \mathcal{F}_n) \\
&= \frac{(n+2)!}{(B_n+1)!(n-B_n)!}\theta^{B_n+1}(1-\theta)^{n-B_n}\frac{B_n+1}{n+2} + \frac{(n+2)!}{B_n!(n+1-B_n)!}\theta^{B_n}(1-\theta)^{n+1-B_n}\frac{n+1-B_n}{n+2} \\
&= \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n} \\
&= N_n
\end{aligned}$$

By taking expectations this implies in particular that the sequence $(\mathbb{E}[N_n])_{n\geq 0}$ is constant, so that $N_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$. Since $\mathbb{E}[N_{n+1}|\mathcal{F}_n] = N_n$, this completes the proof.

□

2 Training exercises

Exercise 2. Let (M_n) be a submartingale such that

$$\sup_{n\geq 1} \mathbb{E}[M_n^+] < \infty$$

where $M_n^+ = \max(M_n, 0)$. Show that $(M_n)_{n\geq 0}$ converges almost surely.

Solution:

Set $C = \sup_{n\geq 1} \mathbb{E}[M_n^+]$. It is enough to show that (M_n) is bounded in L^1. To this end write $M_n = M_n^+ - M_n^-$. Since $(M_n)_{n\geq 1}$ is a submartingale, we have $\mathbb{E}[M_n] \geq \mathbb{E}[M_0]$, so $\mathbb{E}[M_n^-] \leq \mathbb{E}[M_n^+] - \mathbb{E}[M_0] \leq C + \mathbb{E}[M_0]$. As a consequence

$$\mathbb{E}[|M_n|] = \mathbb{E}[M_n^+] + \mathbb{E}[M_n^-] \leq 2C - \mathbb{E}[M_0],$$

which shows that (M_n) is bounded in L^1. □

Exercise 3. Let $(X_i)_{i\geq 1}$ be i.i.d. random variables with values in $\{-1,1\}$ where we write $\mathbb{P}(X_i = 1) = p$ and assume that $p \in (0, 1/2)$. Moreover, define $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. For $n \geq 0$ we set

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}.$$

For $n \geq 1$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

(1) Show that (M_n) is an L^1 bounded martingale with respect to the filtration $(\mathcal{F}_n)$ where.

(2) Show that M_n converges almost surely to 0 as $n \to \infty$, but does not converge in L^1.

Solution:

(1) We observe that (M_n) is $(\mathcal{F}_n)$-measurable and $\mathbb{E}(|M_n|) \leq (1/p-1)^n < \infty$ for all $n \geq 0$. Using that S_n is $\mathcal{F}_n$ measurable and that X_{n+1} is independent of $\mathcal{F}_n$, we get that a.s.

$$\begin{aligned}\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) =& (1/p-1)^{S_n} \mathbb{E}((1/p-1)^{X_{n+1}}) \\ =& M_n \left(p \frac{1-p}{p} + (1-p) \frac{p}{1-p} \right) = M_n.\end{aligned}$$

Therefore, M_n is a martingale, and for all $n \geq 1$, $\mathbb{E}(M_n) = \mathbb{E}(M_0) = 1$, i.e. the sequence (M_n) is bounded in L^1.

(2) Notice that by the strong law of large numbers, S_n/n converges a.s. to $\mathbb{E}(X_1) = 2p-1 < 0$ as $n \to \infty$. This means that a.s. $S_n \to -\infty$ as $n \to \infty$, and thus $M_n \to 0$ a.s. as $n \to \infty$.

Now argue by contraction and assume that $M_n \to M$ in L^1. Then $M_n \to M$ in probability, and since $M_n \to 0$ a.s. we also have $M_n \to 0$ in probability. Thus $M_n \to 0$ in L^1, which is a contradiction since $\mathbb{E}[M_n] = 1$ for every $n \geq 0$.

Remark. Note that for martingales, L^1 convergence is stronger than a.s. convergence, in the sense that a martingale that converges in L^1 also converges a.s. (note that this is not true for general random variables). Indeed, if a martingale converges in L^1, then it is bounded in L^1, so it converges almost surely due to the martingale a.s. convergence theorem. The converse is not true as seen in this exercise.

□

Exercise 4.

(1) Find an example of a martingale which is not bounded in L^1.

(2) Find an example of a martingale which converges almost surely but which is not bounded in L^1.

(3) Find an example of a martingale which converges almost surely to ∞.

Hint. Search for martingales of the form $M_n = X_1 + \cdots + X_n$.

Solution:

In the following, the filtration is taken to be the canonical filtration

(1) Let (M_n) be the simple random walk, defined by $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$ with $(X_i)_{i \geq 1}$ i.i.d. with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$, with its canonical filtration. It is clearly a martingale. If

it is bounded in L^1, then it converges almost surely. Since it is integer valued, this would imply that almost surely it is constant after some point, which is not possible.

(2) Let $(X_n)_{n\geq 1}$ be independent random variables with law given by

$$\mathbb{P}(X_n = 4^n) = \mathbb{P}(X_n = -4^n) = \frac{1}{2^n}, \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{2^{n-1}}$$

and $M_n = X_1 + \cdots + X_n$.

Since $\mathbb{E}[X_n] = 0$ for every $n \geq 1$, (M_n) is a martingale with respect to its canonical filtration. Since $\sum_{n\geq 1} \frac{1}{2^{n-1}} < \infty$, Borel-Cantelli 1 implies that $\mathbb{P}(\limsup\{|X_n| = 4^n\}) = 0$. Thus with probability 0 we have $X_n = 4^n$ infinitely often. In other words, a.s. $X_n = 0$ for n sufficiently large, which implies that a.s. M_n converges.

But if $X_n = 4^n$, then $M_n \geq 4^n - 4^{n-1} - \cdots - 1 \geq 4^{n-1}$, so

$$\mathbb{E}[|M_n|] \geq \mathbb{E}\left[|M_n| \mathbb{1}_{X_n = 4^n}\right] \geq 4^{n-1}\mathbb{P}(X_n = 4^n) = 2^{n-2} \underset{n\to\infty}{\longrightarrow} \infty,$$

so (M_n) is not bounded in L^1.

(3) Let $(X_n)_{n\geq 1}$ be independent random variables with law given by

$$\mathbb{P}(X_n = 1) = \frac{n^2}{n^2+1}, \qquad \mathbb{P}(X_n = -n^2) = \frac{1}{n^2+1}$$

and $M_n = X_1 + \cdots + X_n$. Since $\mathbb{E}[X_n] = 0$ for every $n \geq 1$, (M_n) is a martingale with respect to its canonical filtration. Since $\sum_{n\geq 1} \frac{1}{n^2+1} < \infty$, by Borel-Cantelli 1 we have $\mathbb{P}(\limsup\{X_n = -n^2\}) = 0$. Thus with probability 0 we have $X_n = -n^2$ infinitely often. In other words a.s. $X_n = 1$ for every n sufficiently large, so $M_n \to \infty$ a.s.

□

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let $(Y_n)_{n\geq 0}$ be a sequence of non-negative i.i.d. random variables with $\mathbb{E}(Y_1) = 1$ and $\mathbb{P}(Y_1 = 1) < 1$. For $n \geq 1$ we let $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$, and we set $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

(1) Show that $X_n = \prod_{i=1}^n Y_i$ defines a martingale with respect to $(\mathcal{F}_n)$.

(2) Show that $X_n \to 0$ as $n \to \infty$ a.s.

Hint. You may use the strict Jensen inequality: if $f : \mathbb{R} \to \mathbb{R}$ is strictly convex and X is a non-constant random variable such that X and $f(X)$ are integrable, then $f(\mathbb{E}[X]) < \mathbb{E}[f(X)]$.

Solution:

(1) Clearly X_n is $\mathcal{F}_n$-measurable. In addition, $X_n \geq 0$ and $\mathbb{E}(X_n) = \prod_{i=1}^n \mathbb{E}(Y_i) = 1$ for all $n \geq 1$. Thus $X_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$. Also $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = \prod_{i=1}^n Y_i \cdot \mathbb{E}(Y_{n+1}) = X_n$, which implies that (X_n) is a $(\mathcal{F}_n)$ martingale.

(2) If $\mathbb{P}(Y_1 = 0) > 0$, then since the events $(\{Y_i = 0\})_{i \geq 1}$ are independent, by the second Borel-Cantelli Lemma, a.s. $\{Y_i = 0\}$ happens infinitely many times. This implies that $X_n = 0$ for all n large enough a.s.

Let us now suppose then that $Y_1 > 0$ almost surely. We show that $\ln(X_n) \to -\infty$ as $n \to \infty$ a.s., which will imply the desired result.

First case. $\ln(Y_1)$ is integrable. Then by using the strict concavity of the logarithm we get $\mathbb{E}[\ln Y_1] < \ln \mathbb{E}[Y_1] = 0$. Then by the strong law of large numbers

$$\frac{1}{n}\ln(X_n) = \frac{1}{n}\sum_{i=1}^n \ln(Y_i) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathbb{E}(\ln(Y_1)) < 0.$$

almost surely. Thus $\ln(X_n) \to -\infty$ a.s.

Second case. $\ln(Y_1)$ is not integrable. Then by monotone convergence $\mathbb{E}[\ln \max(Y_1, \epsilon)] \to -\infty$ as $\varepsilon \to 0$, so we can choose $\varepsilon > 0$ such that $\mathbb{E}[\ln \max(Y_1, \epsilon)] < 0$. Then by the strong law of large numbers

$$\frac{1}{n}\ln(X_n) \leq \frac{1}{n}\sum_{i=1}^n \ln(\max(Y_i, \epsilon)) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathbb{E}[\ln \max(Y_1 \vee \epsilon)] < 0$$

Thus $\ln(X_n) \to -\infty$ a.s.

□

Exercise 6. (Bellman's Optimality Principle) Your winnings per unit stake on game n are ϵ_n, where the ϵ_n are i.i.d. random variables with

$$\mathbb{P}(\epsilon_n = +1) = p, \quad \mathbb{P}(\epsilon_n = -1) = q, \quad \text{where } 1/2 < p = 1 - q < 1.$$

Your stake C_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Your objective is to maximize the expected *interest rate* $\mathbb{E}[\ln(Z_N/Z_0)]$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0, is a given constant. Let $\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$ be your *history* up to time n. We assume that $\ln(Z_n)$ is integrable for all $n \geq 0$.

(1) Show that if C is any (predictable) strategy, meaning that for all $n \geq 1$, C_n is $\mathcal{F}_{n-1}$ measurable, then $\ln(Z_n) - n\alpha$ is a supermartingale, where α denotes the *entropy*

$$\alpha = p\ln p + q\ln q + \ln 2.$$

Conclude that $\mathbb{E}[\ln(Z_N/Z_0)] \leq N\alpha$.

(2) Show that for a certain strategy, $\ln(Z_n) - n\alpha$ is a martingale. What is the best strategy?

Solution:

(1) We note that our fortune at time $n+1$ is given by $Z_{n+1} = Z_n + \epsilon_{n+1} C_{n+1}$. Then, (Z_n) is clearly $(\mathcal{F}_n)$ measurable and integrable. Using that Z_n and C_{n+1} are $\mathcal{F}_n$ measurable, we get

$$\begin{aligned}\mathbb{E}[\ln(Z_{n+1}) - \ln(Z_n) \mid \mathcal{F}_n] &= \mathbb{E}[\ln(1 + \epsilon_{n+1} C_{n+1}/Z_n) \mid \mathcal{F}_n] \\ &= p\ln(1 + C_{n+1}/Z_n) + q\ln(1 - C_{n+1}/Z_n)\end{aligned}$$

We observe that the function $p\ln(1+x) + q\ln(1-x)$ is concave and has a maximum at $x = p - q$. Therefore

$$\mathbb{E}[\ln(Z_{n+1}) - \ln(Z_n) \mid \mathcal{F}_n] \leq p\ln(1 + p - q) + q\ln(1 + q - p) = \alpha.$$

This implies that $\mathbb{E}[\ln(Z_{n+1}) - (n+1)\alpha \mid \mathcal{F}_n] \leq \ln(Z_n) - n\alpha$, i.e. $\ln(Z_n) - n\alpha$ is a supermartingale. Therefore,

$$\mathbb{E}[\ln(Z_N/Z_0)] = \mathbb{E}[\ln(Z_N) - N\alpha] - \mathbb{E}(\ln(Z_0)) + N\alpha \leq N\alpha.$$

(2) From the previous part, we know that $\ln(Z_n) - n\alpha$ is a martingale if $C_{n+1} = (2p-1)Z_n$ for all $n \geq 0$. Since the maximum of $p\ln(1+x) + q\ln(1-x)$ is unique, we conclude that this strategy is optimal.

□

Exercise 7. Find a sequence $(M_n)_{n\geq 0}$ of integrable random variables such that $\mathbb{E}[M_{n+1}|M_n] = M_n$ but such that $(M_n)_{n\geq 0}$ is not a martingale with respect to its canonical filtration.

Solution:
Take a simple random walk, that is $M_n = X_1 + \cdots + X_n$ with $(X_i)_{i\geq 1}$ i.i.d. with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$, but slightly modified so that when it hits 0 it repeats the step X_1. Indeed, for $n \geq 1$, we have

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \begin{cases} M_n & \text{if } M_n \neq 0 \\ -1 & \text{if } M_n = 0 \text{ and } M_1 = -1 \\ 1 & \text{if } M_n = 0 \text{ and } M_1 = 1. \end{cases}$$

In paricular, $\mathbb{E}[M_n|\mathcal{F}_n] \neq M_n$ if $M_n = 0$. It is not too difficult to see that a.s. there exists $n \geq 1$ with $M_n = 0$. Thus (M_n) is not a martingale. □

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 8. A mathematician, an economist and a trader are chatting in a bar. The economist says:

"The euro value of a CHF over time is a martingale! Otherwise, it would be possible to make money on average, buying and selling CHF at the right time!"

The mathematician replies: "But if that's true, according to conditional's Jensen's inequality, the CHF value of a euro is a sub-martingale!"

The trader says nothing, thinks for a few seconds, then runs off to buy euros.

What do you think?

Solution:

To begin with, the mathematician is of course right. Let $(M_n)_{n\geq 0}$ be a positive martingale for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Since the inverse function is convex on $\mathbb{R}_+^*$, Jensen's conditional inequality gives

$$\mathbb{E}[M_{n+1}^{-1}|\mathcal{F}_n] \geq \mathbb{E}[M_{n+1}|\mathcal{F}_n]^{-1} = M_n^{-1},$$

so $(M_n^{-1})_{n\geq 0}$ is indeed a submartingale. Moreover, if M_{n+1} is not $\mathcal{F}_n$ -measurable (which must be the case the case in practice), then the inequality is strict. Finally, if M_n is the euro value of a CHF, then M_n^{-1} is the CHF value of an euro, so the euro/CHF rate would be a strict sub-martingale.

The trader's reasoning is as follows. If the euro/CHF rate is a strict sub-martingale, then by buying euros today and selling them tomorrow, the expected gain in CHF is strictly positive. So you have to do it! On the other hand, this reasoning seems bizarre: it suggests that it would be systematically interesting to buy euros and sell them the next day, whereas the same reasoning would lead to the opposite if we swapped the roles of the euro and the CHF. This means that the economist's assertion must be false: the rates euro/CHF and CHF/euro would both be sub-martingales! This would mean that buying euros one day and selling them the next would result in a positive expectation of gains in CHF. So once again, we have the impression that we can win every time!

One can explain this "paradox" as follows. We win CHF when the rate of CHF fells: thus, if we win CHF then each of these CHF is worth less than at the beginning. We therefore have a positive gain in CHF, but once the CHF have been converted into goods, the expectation of gain should become zero. □