Recap on sets

1 Operations on sets

Let E be a non-empty set.

Subsets. We write $A \subset E$ and say that A is a subset of E if for any $x \in A$, we have $x \in E$.

Complementary. If $A \subset E$, let $A^c = \{x \in E : x \notin A\}$ be the complement of A in E (A^c is a set consisting of elements of E that are not in A). If $A \subset B$, let $B \backslash A$ denote the elements of B that do not belong to A. Then $B \backslash A = B \cap (A^c)$.

Union. Let I be any set (we see I as a set of indices), and let $(A_i)_{i \in I}$ be a collection of subsets of E. Then $\bigcup_{i \in I} A_i$ denotes the subset of E formed by the elements x such that there exists $i \in I$ with $x \in A_i$.

Intersection. Let I be any set (we see I as a set of indices), and let $(A_i)_{i \in I}$ be a collection of subsets of E. Then $\bigcap_{i \in I} A_i$ denotes the subset of E formed of elements x such that for any $i \in I$ we have $x \in A_i$.

Unions, intersections and complements. Let I be any set (we see I as a set of indices), and let $(A_i)_{i \in I}$ be a collection of subsets of E. Recall that:

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} (A_i)^c, \qquad \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} (A_i)^c.$$

Equality of two sets. If A and B are two sets, to show that $A = B$, we very often reason by double inclusion, showing that if $x \in A$, then $x \in B$, then that if $x \in B$, then $x \in A$.

Set of parts. Let $\mathcal{P}(E)$ be the set of subsets of E (also called the power set of E).

Cartesian product of sets. If $E_1, \ldots, E_n$ are sets, let $E_1 \times \cdots \times E_n$ be the set of n-uplets $(x_1, \ldots, x_n)$ such that $x_1 \in E_1, x_2 \in E_2, \ldots, x_n \in E_n$. If I is a set, let E^I be the set of all applications of I in E. An element of E^I is usually denoted by $(e_i)_{i \in I}$.

2 Direct image, preimage

Let X and Y be two sets. Let $f : X \to Y$ be an application.

(∗) **Direct image.** If $A \subset X$ is a subset of X, let $f(A)$ be the **subset** of Y defined by

$$f(A) = \{y \in Y : \text{there exists } x \in A \text{ such that } y = f(x)\}.$$

We sometimes write $f(A) = \{f(x) : x \in A\}$.

(∗) **Preimage.** If $B \subset Y$ is a subset of Y, let $f^{-1}(B)$ be the **subset** of X defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

We have $f(x) \in B$ if and only $x \in f^{-1}(B)$. We have $f(A) \subset B$ if and only if $A \subset f^{-1}(B)$.

WARNING. If $y \in Y$, the notation $f^{-1}(y)$ doesn't always make sense (unless f is injective), whereas $f^{-1}(\{y\})$ does. On the other hand, if $x \in X$, $f(x)$ always makes sense (it's an element of Y), as does $f(\{x\})$ (which is a subset of Y that is $\{f(x)\}$).

(∗) **Composition of preimages.** Let X, Y, Z be sets and $f : X \to Y$, $g : Y \to Z$ be applications. Then for any subset $B \subset Z$ of Z we have

$$(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B)),$$

which are subsets of X.

3 = 1+2

Let $f : X \to Y$ be an application. Let I be any set (we see I as a set of indices), let $(A_i)_{i\in I}$ be a collection of subsets of X and let $(B_i)_{i\in I}$ be a collection of subsets of Y. Then

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$$

and

$$f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i), \qquad \boxed{f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)}.$$

Furthermore, if $A \subset B \subset Y$, we have $f^{-1}(B\backslash A) = f^{-1}(B)\backslash f^{-1}(A)$.

WARNING. In general, it's not true that $f\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$ (find a counterexample!).