Approximation toolbox

The following techniques are useful:

(1) Writing a real-valued random variable Z as $Z = Z^+ - Z^-$ with $Z^+, Z^- \geq 0$. This often allows to easily get that a property holds for real-valued random variables after showing that it holds for non-negative ones (typically when we have linearity in Z).

(2) Writing a non-negative real-valued random variable $Z \geq 0$ as a pointwise increasing limit $0 \leq Z_n \uparrow Z$ of random variables. This often allows to easily get that a property holds for non-negative random variables after showing it holds for simple non-negative random variables (typically using monotone convergence).

(3a) Writing a real-valued random variable Z as a pointwise limit of bounded functions $Z = \lim_{n\to\infty} Z\mathbb{1}_{|Z|\leq n}$. This often allows to easily get that a property holds for real-valued random variables after showing it holds for bounded ones (typically using dominated convergence)

(3b) Writing a non-negative real-valued random variable $Z \geq 0$ as a pointwise limit of bounded functions $Z = \lim_{n\to\infty} Z\mathbb{1}_{|Z|\leq n}$. This often allows to easily get that a property holds for non-negative real-valued random variables after showing it holds for non-negative bounded ones (typically using monotone convergence).

(4) When $\mathcal{A}$ is a σ-field, instead of considering $\mathbb{1}_A$ with $A \in \mathcal{A}$, considering $\mathbb{1}_C$ with C in a generating π-system of $\mathcal{A}$. This sometimes allows to show that a property holds for simple functions after showing that it holds for simple ones with indicators of measurable sets in an explicit "nice" generating π-system (typically using linearity and the Dynkin Lemma).