Submartingales, supermartingales, martingales

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a filtration $(\mathcal{F}_n)_{n\geq 0}$ of $\mathcal{F}$ ($\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$).

1 Definitions

– A sequence of random variables $(M_n)_{n\geq 0}$ is a *martingale* if for every $n \geq 0$, $M_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ and $\mathbb{E}[M_{n+1} \mid F_n] = M_n$.

– A sequence of random variables $(M_n)_{n\geq 0}$ is a *submartingale* if for every $n \geq 0$, $M_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ and $\mathbb{E}[M_{n+1} \mid F_n] \geq M_n$.

– A sequence of random variables $(M_n)_{n\geq 0}$ is a *supermartingale* if for every $n \geq 0$, $M_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ and $\mathbb{E}[M_{n+1} \mid F_n] \leq M_n$

$\ast$ Intuitively, a martingale represents a fair game, a submartingale a favorable game (“submartingales tend to increase”) and a supermartingale represents a defavorable game (“supermartingales tend to decrease”).

More formally, the sequence $(\mathbb{E}[M_n])_{n\geq 0}$:

– is constant for a martingale

– is weakly increasing for a submartingale

– is weakly decreasing for a supermartingale

2 Convergence theorems

Almost sure convergence. A martingale, supermartingale or submartingale bounded in L^1 converges almost surely.
WARNING. the converse is false in general: a (sub/super)martingale can converge almost surely without being bounded in L^1.

L^1 **convergence.** A martingale, supermartingale or submartingale converges in L^1 if and only if it is uniformly integrable (indeed, convergence in L^1 always implies uniformly integrability, and uniform integrability implies bounded in L^1, thus a.s. convergence for a (sub/super)martingale, thus convergence in probability and thus convergence in L^1 by uniform integrability). In these cases, they also converge a.s.

In the case of a martingale (M_n), this is equivalent to the fact that (M_n) is a closed martingale in the form $M_n = \mathbb{E}[M_\infty \mid \mathcal{F}_n]$ where M_∞ is the a.s. limit of (M_n).

L^p **convergence for** $p > 1$**.** A martingale converges in L^p if and only if it is bounded in L^p.
WARNING. This is not true for submartingales or supermartingales.

3 Optional stopping

The optional stopping theorem gives conditions under which $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ for a martingale (M_n) and a random time T.

Optional stopping theorem. When T is a stopping time (i.e. for every $n \geq 0$ we have $\{T \leq n\} \in \mathcal{F}_n$), and when (M_n) is a uniformly integrable martingale, we have $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

In practice we often apply the optional stopping theorem with a stopping T which is finite almost surely and by checking that the stopped martingale $(M_{n\wedge T})_{n\geq 0}$ is uniformly integrable (it then converges a.s. and in L^1 to M_T, so $\mathbb{E}[M_0] = \mathbb{E}[M_T]$).

In practice, to show uniform integrability, we often use the fact that a bounded sequence of random variables is uniformly integrable, and more generally that a sequence of random variables bounded in L^p with $p > 1$ is uniformly integrabe.

4 Some terminology

Various notions of boundedness are involved; let us recall them.

Finiteness of a random variable. A $\mathbb{R} \cup \{\infty\}$-valued random variable X is said to be:

- *finite* if $|X| < \infty$ (that is for every $\omega \in \Omega$ we have $|X(\omega)| < \infty$);
- *almost surely finite* if almost surely $|X| < \infty$ (that is the set of $\omega \in \Omega$ such that $|X(\omega)| < \infty$ has probability 1).

In practice, sometimes one uses "finite" instead of "almost surely finite" (as in conditional expectations the "a.s." is implicit).

Boundedness of a random variable. A $\mathbb{R}^d$-valued random variable X is said to be:

- *bounded* if there exists $M > 0$ such that $|X| \leq M$ (that is for every $\omega \in \Omega$ we have $|X(\omega)| \leq M$);
- *almost surely bounded* if there exists $M > 0$ such that almost surely $|X| \leq M$ (that is the set of $\omega \in \Omega$ such that $|X(\omega)| \leq M$ has probability 1).

In practice, sometimes uses "bounded" instead of "almost surely bounded" (as in conditional expectations the "a.s." is implicit).

WARNING. Here and after it is crucial that M does not depend on ω.

Boundedness of a sequence of random variables. A sequence $(X_n)_{n\geq 1}$ of $\mathbb{R}^d$-valued random variables is said to be:

- *bounded* if there exists $M > 0$ such that for every $n \geq 1$ we have $|X_n| \leq M$ (that is for every $\omega \in \Omega$ we have $|X_n(\omega)| \leq M$).
- *almost surely bounded* if there exists $M > 0$ such that almost surely for every $n \geq 1$ we have $|X_n| \leq M$ (that is for every $\omega \in \Omega$ we have $|X_n(\omega)| \leq M$).

WARNING. A sequence of bounded random variables is not necessarily a bounded sequence of random variables! For example, if $X_n = n$, then for every $n \geq 1$ the random variable X_n is bounded, but the sequence $(X_n)_{n\geq 1}$ is not bounded.

Remark. Since one can interchange "a.s." and "for every on a countable set", observe that "almost surely for every $n \geq 1$ we have $|X_n| \leq M$" is equivalent to "for every $n \geq 1$ almost surely we have $|X_n| \leq M$".

Boundedness in $\mathbb{L}^p$ of a sequence of random variables. A sequence $(X_n)_{n\geq 1}$ of $\mathbb{R}$-valued random variables is said to be:

- *bounded in* $\mathbb{L}^p$ if $\sup_{n\geq 1} \mathbb{E}\left[|X_n|^p\right] < \infty$ (that is there exists $M > 0$ such that $\mathbb{E}\left[|X_n|^p\right] \leq M$ for every $n \geq 1$).