Recap on probability spaces

$*$ **Probabilized spaces.** $(\Omega, \mathcal{A}, \mathbb{P})$ is a probabilized space (sometimes also called a probability space) if $\mathcal{A}$ is a σ-field on Ω and $\mathbb{P}$ is a probability (sometimes also called a probability measure) on $(\Omega, \mathcal{A})$.

$*$ **σ-fields** A set $\mathcal{A}$ of subsets of Ω ($\mathcal{A} \subset \mathcal{P}(\Omega)$) is a σ-field on Ω if :

(1) $\Omega \in \mathcal{A}$.

(2) For all $A \in \mathcal{A}$, we have $A^c = \Omega \backslash A \in \mathcal{A}$.

(3) For any sequence $(A_n) \in \mathcal{A}^{\mathbb{N}}$ of elements of $\mathcal{A}$, we have $\bigcup_{n \geq 0} A_n \in \mathcal{A}$.

The elements of $\mathcal{A}$ are said to be the *events*.
☡ WARNING. An event is **always** a subset of Ω.

$*$ **Probability** A probability $\mathbb{P}$ is a

$$\textbf{application} \quad \mathbb{P} : \mathcal{A} \to [0,1]$$

such that

(1) $\mathbb{P}(\Omega) = 1$

(2) For any sequence $(A_n) \in \mathcal{A}^{\mathbb{N}}$ of pairwise disjoint $\mathcal{A}$ events, we have

$$\mathbb{P}\left(\bigcup_{n \geq 0} A_n\right) = \sum_{n=0}^{\infty} \mathbb{P}(A_n).$$

Probabilistic modeling consists in describing an a priori random experiment by making the choice of a probability space.

$*$ **Independence** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let I be a set. Events $(A_i)_{i \in I}$ are independent (implying **mutually** and relative to $\mathbb{P}$) if for any finite number of indices $i_1, i_2, \ldots, i_n$ we have

$$\mathbb{P}\left(\bigcap_{1 \leq k \leq n} A_{i_k}\right) = \prod_{k=1}^{n} \mathbb{P}(A_{i_k}).$$

Or, written another way, $\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_n})$.
☡ WARNING. The notion of independence of a sequence of events is very strong: it involves many equality conditions (one for each finite subset of I).

$*$ **The rules of the game.**

(a) $\mathbb{P}(\emptyset) = 0$.

(b) For all $A \in \mathcal{A}$, we have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(c) If $A, B \in \mathcal{A}$ and $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$ and $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$.

(d) If $A, B \in \mathcal{A}$, we have $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ (generalization: sieve formula).

$*$ **Probabilities as limits.** Let $(A_n) \in \mathcal{A}^{\mathbb{N}}$ be a sequence of events.

(e) We have $\mathbb{P}\left(\bigcup_{n \geq 0} A_n\right) \leq \sum_{n \geq 0} \mathbb{P}(A_n)$.

(f) **(Increasing union)** If (A_n) is **increasing for inclusion** (i.e. $A_k \subset A_{k+1}$ for all $k \geq 0$), then $\mathbb{P}(A_n) \to \mathbb{P}\left(\bigcup_{n \geq 0} A_n\right)$ when $n \to \infty$.

(g) **(Decreasing intersection)** If (A_n) is **decreasing for inclusion** (i.e. $A_{k+1} \subset A_k$ for all $k \geq 0$), then $\mathbb{P}(A_n) \to \mathbb{P}\left(\bigcap_{n \geq 0} A_n\right)$ when $n \to \infty$.