Recap on countable sets

∗ **Countable/at most countable/uncountable sets.** Denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of nonnegative integers. A set E is said to be *countable* if there exists a bijection $\phi : \mathbb{N} \to E$. It is sometimes convenient to call a set *at most countable* if it is finite or countable, and to call a set *uncountable* if it is not at most countable (which means that the set is infinite and not countable).

Thus, writing $x_n = \phi(n)$, we have $E = \{x_n; n \geq 0\}$ (we sometimes say that we describe E in extension). Equivalently, we can also write $E = \{x'_n; n \geq 1\}$ (setting $x_{n+1} = x'_n$). We sometimes also say that we are enumerating the elements of E.

∗ **Countable sets and functions.** The following results are useful to show that sets are countable or uncountable by using one-to-one functions or onto functions: Let E, F be two sets and

$$f : E \to F$$

a function.

(1) if E is countable and f is onto, then F is countable.

(2) if F is countable and f is one-to-one, then E is at most countable.

(3) if f is a bijection, E is countable if and only if F is countable.

The contrapositive statements of (1) and (2) yield:

(4) if F is uncountable and f is onto, then E is uncoutable.

(5) if E is uncountable and f is one-to-one, then F is uncountable

∗ **Examples.**

(i) $\mathbb{Q}$ is countable (if a rational is written in the form p/q with $\gcd(p, q) = 1$, the function defined by $0 \mapsto 0$ and $p/q \mapsto 2^p(2q+1)$ is a one-to-one function from $\mathbb{Q}$ to $\mathbb{N}$, so $\mathbb{Q}$ is countable by (2)).

(ii) $\{0, 1\}^{\mathbb{N}}$ is uncountable. Indeed, argue by contradiction, and assume that $\{0, 1\}^{\mathbb{N}} = \{x^k : k \geq 0\}$. Write $x^k = (x^k_n)_{n\geq 0}$ and consider the sequence $y = (y_n)_{n\geq 0}$ defined by $y_n = 1$ if $x^n_n = 0$ and $y_n = 0$ if $x^n_n = 1$. Then there exists $k \geq 0$ such that $x^k = y$. But then $x^k_k = y_k$ and $x^k_k \neq y_k$ by construction, which is absurd (this is the so-called Cantor diagonal argument).

(iii) $\mathbb{R}$ is uncountable (the map $f : \{0, 1\}^{\mathbb{N}} \to \mathbb{R}$ defined by $f((x_n)_{n\geq 0}) = \sum_{n=0}^{\infty} x_n 10^{-n}$ is one-to-one, so $\mathbb{R}$ in uncountable by (5))

∗ **Useful properties.**

(a) Let E be a countable set and $A \subset E$ be a subset. Then A is at most countable.

(b) Let I be an at most countable set, and for every $i \in I$ consider an at most countable set A_i. Then the set

$$\bigcup_{i\in I} A_i$$

is at most countable.

Usually, one says that "a countable union of countable sets is countable".

(c) Let $k \geq 1$ be an integer and for every $i \in \{1, 2, \ldots, k\}$ consider an at most countable set A_i. Then the set

$$A_1 \times A_2 \times \cdots \times A_k$$

is at most countable.

WARNING. An countable product of countable sets is not necessarily countable: $\{0, 1\}^{\mathbb{N}}$ is uncountable (see (ii) above).

∗ **Application.** The set E of polynomials with rational coefficients is countable. Indeed, let E_n be the set of polynomials with rational coefficients of degree n. Since a polynomial of degree n has $n + 1$ coefficients, E_n is in bijection with $\mathbb{Q}^{n+1}$, which is countable as a finite product of countable sets. Then

$$E = \bigcup_{n\geq 0} E_n$$

is countable as a countable union of countable sets.