Recap on some topological aspects

If (E, d) is a metric space, the open ball centered at $x \in E$ and with radius $r > 0$ is $B(x, r) = \{y \in E : d(x, y) < r\}$. Sometimes we write $B_E(x, r)$. We denote by $\mathcal{B}(E)$ or $\mathcal{B}_E$ the Borel σ-field on E, which the σ-field generated by all open sets of E. Recall that $A \subset E$ is open if for every $x \in A$ there exists $r > 0$ such that $B(x, r) \subset A$, and that *any* union (possibly uncountable) is open.

Observe that in $\mathbb{R}$ an open interval is an open ball (take the center in the middle of the interval).

1 $\mathbb{R}^n$.

$*$ $\mathbb{R}$. In $\mathbb{R}$, every open set can be written as an at most countable union of disjoint open intervals.

Proof. Indeed, if O is an open set and $x \in O$, then there exists an open interval I such that $x \in I \subset O$. If there exists one such interval, then there exists a maximal interval which contains x (the union of all such open intervals, which is open as a union of open sets). Denote by $(O_i)_{i\imath I}$ the family of such maximal intervals. First, all intervals O_i are pairwise disjoint (otherwise they wouldn't be maximal). Second, any interval contains a rational number, so one can construct a one-to-one map from I to $\mathbb{Q}$. This shows that I is at most countable. □

$*$ $\mathbb{R}^n$. In $\mathbb{R}^n$, every open set can be written as an at most countable union of open balls. As a corallary, the Borel σ-field on $\mathbb{R}^n$ is generated by open balls.

Proof. Consider an open set $A \subseteq \mathbb{R}^n$. Then for every $x \in A$, there is some $\epsilon_x > 0$ such that the open ball $B(x, 2\epsilon_x)$ is contained in A. We may assume that ϵ_x is rational by making it smaller. Because $\mathbb{Q}^n$ is dense, we can pick a point $q_x \in \mathbb{Q}^n$ with $\|x - q_x\| < \epsilon_x$. In particular $q_x \in A$. Let us show that

$$A = \bigcup_{x \in A} B(q_x, \epsilon_x) \tag{1}$$

by double inclusion:

– Take $x \in A$. Since $x \in B(q_x, \epsilon_x)$, we get $A \subseteq \bigcup_{x \in A} B(q_x, \epsilon_x)$.

– For the reverse inclusion, observe that for every $x \in A$ we have $B(q_x, \epsilon_x) \subseteq A$. Therefore, $\bigcup_{x \in A} B(q_x, \epsilon_x) \subseteq A$.

This shows (1). Finally observe that the union in (1) is a countable union, because there are only countably many balls with rational center and rational radius. □

WARNING. However, it is NOT true that in $\mathbb{R}^n$, every open set can be written as an at most countable *disjoint* union of open balls. For example, one can show that $\mathbb{R}^2 \backslash \{(0, 0)\}$ is not a countable disjoint union of open balls.

WARNING. In general, it is not true in general that in any metric space every open set can be written as an at most countable union of open balls (it is true for *separable* metric spaces, which are metric spaces that admit a countable dense sequence, such as $\mathbb{R}^n$ for example). In particular, it is not true in general that the Borel σ-field of a metric space is generated by open balls (but it is generated by open sets, by definition).

2 Induced topology, induced σ-field

Let (E,d) be a metric space and $A \subset E$. We view A as a metric space with distance d.

$*$ The following results gives the form of all open sets of (A,d) and of all elements of the Borel σ-field of (A,d) in terms of open sets of E and elements of the Borel σ-field of (E,d).

(1) The open sets of (A,d) are of the form $A \cap O$ with O open set in (E,d).

(2) The elements of $(A,\mathcal{B}(A))$ are of the form $A \cap B$ with $B \in \mathcal{B}(E)$.

Proof.

(1) The inclusion $I : (A,d) \to (E,d)$ defined by $I(x) = x$ being continuous, the preimage of any open is an open, so elements of the form $I^{-1}(O) = A \cap O$ with O open in (E,d) are open in (A,d).

Conversely, let $U \subset A$ be open in (A,d). For every $x \in U$, there exists $r_x > 0$ such that $B_A(x,r_x) \subset U$. Then

$$U = \bigcup_{x \in U} B_A(x,r_x).$$

Now $B_A(x,r_x) = B_E(x,r_x) \cap A$. Therefore

$$U = \bigcup_{x \in U} (B_E(x,r_x) \cap A) = \left(\bigcup_{x \in U} B_E(x,r_x) \right) \cap A,$$

and the result follows by taking $O = \cup_{x \in U} B_E(x,r_x)$, which is open in E as a union of open sets.

(2) Since the inclusion $I : (A,d) \to (E,d)$ is continuous and therefore measurable, we deduce that elements of the form $A \cap B$ with $B \in \mathcal{B}(E)$ are elements of $(A,\mathcal{B}(A))$.

Conversely, elements of the form $A \cap B$ with $B \in \mathcal{B}(E)$ contain the open of A (from the previous question) and form a tribe, so contain elements of $(A,\mathcal{B}(A))$. □

WARNING. Observe that A is not necessarily an element of the Borel σ-field of E! It still makes sense to consider $\mathcal{B}(A)$.