Recap on random variables

1 Random Variables

Let $(\Omega, \mathcal{A})$ and $(E, \mathcal{E})$ be two probability spaces. A **random variable** from Ω to E is a function $X : \Omega \to E$ that is measurable (meaning $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{E}$).

WARNING. The definition of a random variable does not involve the probability $\mathbb{P}$.

When say a "random variable taking values in E," we mean a random variable with the implied sigma-algebra on E (when $E = \mathbb{R}$, the Borel sigma-algebra is usually taken), and the underlying probability space is also implied.

2 "Probabilistic" notation

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ be a random variable. If $B \in \mathcal{E}$:

- $\{X \in B\}$ is a *notation* for the event $\{\omega \in \Omega : X(\omega) \in B\}$, used for simplification (it is also sometimes denoted by $(X \in B)$, but we often prefer to write $\{X \in B\}$ to emphasize that $\{X \in B\}$ is a set, namely a subset of Ω belonging to $\mathcal{A}$).
- $\{X \in B\} = X^{-1}(B)$ by definition of the inverse image (and $X^{-1}(B) \in \mathcal{A}$ since X is a random variable).
- $\mathbb{P}(X \in B)$ is a *notation* for $\mathbb{P}(\{X \in B\})$, used for simplification.

In the same vein, if $Y : (\Omega, \mathcal{A}) \to (F, \mathcal{F})$ is another random variable and $C \in \mathcal{F}$:

- $\mathbb{P}(X \in B, Y \in C)$ and $\mathbb{P}(X \in B$ and $Y \in C)$ both mean $\mathbb{P}(\{X \in B\} \cap \{Y \in C\})$, or equivalently $\mathbb{P}(\{\omega \in \Omega : X(\omega) \in B$ and $Y(\omega) \in C\})$.
- $\mathbb{P}(X \in B$ or $Y \in C)$ means $\mathbb{P}(\{X \in B\} \cup \{Y \in C\})$, or equivalently $\mathbb{P}(\{\omega \in \Omega : X(\omega) \in B$ or $Y(\omega) \in C\})$.

WARNING. In the two previous points, X and Y do not necessarily have to take values in the same space, but they must be defined on the same underlying space.

For example, $\mathbb{P}(X \geq 2)$ is a notation for $\mathbb{P}(\{\omega \in \Omega : X(\omega) \geq 2\})$, and $\mathbb{P}(X = Y)$ is a notation for $\mathbb{P}(\{\omega \in \Omega : X(\omega) = Y(\omega)\})$.

3 Laws of Random Variables

If $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ is a random variable and $\mathbb{P}$ is a probability on $(\Omega, \mathcal{A})$, the **law of X under $\mathbb{P}$** is a

probability *on the target space (co-domain)* $(E, \mathcal{E})$,

often denoted by $\mathbb{P}_X$, defined by

$$\text{for all } B \in \mathcal{E}, \quad \mathbb{P}_X(B) \quad := \quad \mathbb{P}(X^{-1}(B)) \quad = \quad \mathbb{P}(X \in B).$$

WARNING. The law of X depends on the probability considered on $(\Omega, \mathcal{A})$. However, if there is only one probability involved on $(\Omega, \mathcal{A})$ and there is no ambiguity, we simply refer to the law of X.

4 Generated σ-fields

If $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ is a random variable, the σ-field generated by X, denoted by $\sigma(X)$, is the smallest sigma field on Ω for which X is measurable (since X is measurable for $\mathcal{A}$ by definition of a random variable, we have $\sigma(X) \subset \mathcal{A}$). We have

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{E}\} \tag{1}$$

Indeed, if X is measurable, that all the elements of the form $X^{-1}(B)$, $B \in \mathcal{E}$ should be in any σ-field for which X is measurable. Since they form a σ-field, we have equality.

If $X_i : (\Omega, \mathcal{A}) \to (E_i, \mathcal{E}_i)$ are random variables, the σ-field generated by the family $(X_i)_{i \in I}$, denoted by $\sigma(X_i : i \in I)$ is the smallest σ-field on Ω for which all the X_i for $i \in I$ are measurable.

WARNING. In general, the σ-field generated by several random variables is **not** explicit:

$$\sigma(X_i : i \in I) = \sigma\left(\{X_i^{-1}(B_i) : B_i \in \mathcal{E}_i, i \in I\}\right). \tag{2}$$

This is in contrast to the case where we have one random variable (1).

WARNING. The second σ (on the right-hand side of the equality (2)) refers to the σ-field generated by a collection of subsets of Ω: recall that if $\mathcal{C} \subset \mathcal{P}(\Omega)$ is a collection of subsets of Ω then $\sigma(\mathcal{C})$ is the smallest sigma-field on Ω containing all elements of $\mathcal{C}$.

Formula (2) gives us nonetheless an explicit generating π-system of $\sigma(X_i : i \in I)$, which is the collection of sets of the form

$$X_{i_1}^{-1}(B_{i_1}) \cap \cdots \cap X_{i_k}^{-1}(B_{i_k})$$

for $k \geq 1$, $i_j \in I$ and $B_{i_j} \in \mathcal{E}_{i_j}$ for $1 \leq j \leq k$.

5 Independent Random Variables

If $X_1 : (\Omega, \mathcal{A}) \to (E_1, \mathcal{E}_1), \ldots, X_n : (\Omega, \mathcal{A}) \to (E_n, \mathcal{E}_n)$ are random variables, we say they are (mutually) independent (with respect to $\mathbb{P}$) if the σ-fields $\sigma(X_1), \ldots, \sigma(X_n)$ are independent, that is:

$$\text{for all } B_1 \in \mathcal{E}_1, \ldots, B_n \in \mathcal{E}_n, \quad \mathbb{P}(X_1 \in B_1, \ldots, X_n \in B_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i).$$

If I is any set, we say that the random variables $(X_i)_{i \in I}$ are independent if, for any $J \subset I$ with $\mathrm{Card}(J) < \infty$, the random variables $(X_j)_{j \in J}$ are independent.

(Informally, independence transforms intersections into products).

WARNING. Random variables can be independent with respect to one probability measure but not independent with respect to another probability measure.