Recap on convergence theorems

All the functions are defined on a measured space $(E, \mathcal{E}, \mu)$. All the random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Recall that a function $f : (E, \mathcal{E}) \to \mathbb{R}$ is said to be integrable (with respect to μ) if $\int |f| d\mu < \infty$, in which case $\int f d\mu$ is equal, by definition, to $\int f^+ f d\mu - \int f^- d\mu$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

1 Monotone convergence theorem

Theorem (Monotone convergence theorem)**.**

Let (f_n) be a sequence of increasing measurable functions with values in $[0, \infty]$. Then

$$\int (\lim_n f_n) d\mu = \lim_{n \to \infty} \uparrow \int f_n d\mu.$$

Theorem (Monotone convergence theorem: probabilistic version)**.**

Let (X_n) be a sequence of increasing random variables with values in $[0, \infty]$. Then

$$\mathbb{E}\left[\lim_n X_n\right] = \lim_{n \to \infty} \uparrow \mathbb{E}\left[X_n\right].$$

2 Fatou's lemma

Theorem (Fatou's lemma)**.**

Let (f_n) be a sequence of **nonnegative** measurable functions. Then

$$\int (\liminf_{n \to \infty} f_n) d\mu \leq \liminf_{n \to \infty} \int f_n d\mu.$$

Theorem (Fatou's lemma: probabilistic version)**.**

Let (X_n) be a sequence of **nonnegative** random variables. Then

$$\mathbb{E}\left[\liminf_{n \to \infty} X_n\right] \leq \liminf_{n \to \infty} \mathbb{E}\left[X_n\right].$$

3 Dominated convergence theorem

Recall that a property is said to be true μ almost everywhere (or simply almost everywhere when there is no ambiguity) if is is true outside a set of 0 measure. For exemple, saying that $f(x) = g(x)$ for μ almost x is saying that $\mu(\{x \in E : f(x) \neq g(x)\}) = 0$.

Theorem (Dominated convergence theorem)**.**

Let (f_n) be a sequence of integrable real-valued measurable functions. Assume that:

(1) there exists a measurable function f such that $f_n(x) \to f(x)$ holds for μ almost x

(2) there exists a measurable function $g : E \to \mathbb{R}_+$ such that $\int g d\mu < \infty$ and for every n, the inequality $|f_n| \leq g$ holds μ almost everywhere.

Then

$$\lim_{n \to \infty} \int |f_n - f| d\mu \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \int f_n d\mu \underset{n\to\infty}{\longrightarrow} \int f d\mu.$$

In the previous result, it is crucial that g does not depend on n.

Theorem (Dominated convergence theorem: probabilistic version)**.**

Let (X_n) be a sequence of integrable real-valued random variables. Assume that:

(1) there exists a random variable X such that $X_n \to X$ holds almost surely

(2) there exists a nonnegative random variable Z such that $\mathbb{E}[|Z|] < \infty$ and for every n, the inequality $|X_n| \leq Z$ holds almost surely.

Then

$$\lim_{n \to \infty} \mathbb{E}[|X_n - X|] \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \mathbb{E}[X_n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[X].$$

4 Fubini's theorems

Here, $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ are measured spaces. Recall that $\mathcal{E} \otimes \mathcal{F}$ is the product σ-field $\sigma(A \times B : A \in \mathcal{E}, B \in \mathcal{F})$. We assume that μ and ν are σ-finite measure, and recall that $\mu \otimes \nu$ is the unique measure on $(\mathcal{E} \otimes \mathcal{F})$ such that $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$ for $A \in \mathcal{E}$ and $B \in \mathcal{F}$ (with the standard convention $0 \times \infty = 0$).

Theorem (Fubini-Tonnelli)**.**

Let $f : E \times F \to [0, \infty]$ be measurable. Then $x \mapsto \int f(x,y)\nu(dy)$ and $y \mapsto \int f(x,y)\mu(dx)$ are both respectively $\mathcal{E}$-measurable and $\mathcal{F}$-measurable, and

$$\int_{E \times F} f d\mu \otimes \nu = \int_E \left(\int_F f(x,y)\nu(dy) \right) \mu(dx) = \int_F \left(\int_E f(x,y)\mu(dx) \right) \nu(dy).$$

In practice, Fubini-Tonnelli says that for **nonnegative** functions (and σ-finite measures), one can always permute the integrals (without changing the result).

Theorem (Fubini-Lebesgue)**.**

Let $f : E \times F \to \mathbb{R}$ be an integrable function with respect to $\mu \otimes \nu$. Then

(1) for μ-almost every x, $y \mapsto f(x,y)$ is integrable with respect to ν and for ν-almost every y, $x \mapsto f(x,y)$ is integrable with respect to μ;

(2) the functions $x \mapsto \int f(x,y)\nu(dy)$ and $y \mapsto f(x,y)\mu(dx)$ are well defined, except maybe on sets with 0 measure, and are respectively integrable with respect to μ and ν;

(3) we have

$$\int_{E\times F} f d\mu \otimes \nu = \int_E \left(\int_F f(x,y)\nu(dy) \right) \mu(dx) = \int_F \left(\int_E f(x,y)\mu(dx) \right) \nu(dy).$$

In practice, to permute the integrals for real-valued functions, one first checks that the integral is finite when taking the absolute value (often by using Fubini-Tonnelli's theorem).