Notions of convergence for random variables

1 Convergence of random variables defined on the same probability space

Let $(X_n)_{n\geq 1}$ be a sequence of random variables defined on the **same** probability space, with values in $\mathbb{R}^n$ equipped with the Borel σ-field. We denote by $|\cdot|$ any norm on $\mathbb{R}^n$ (take for example the standard Euclidean $\|\cdot\|_2$ norm).

Almost sure convergence. We say that X_n converges almost surely to X if with probability 1, X_n converges to X as $n \to \infty$, that is if $\mathbb{P}\left(X_n \underset{n\to\infty}{\longrightarrow} X\right) = 1$, or, equivalently,

$$\mathbb{P}\left(\{\omega \in \Omega : X_n(\omega) \underset{n\to\infty}{\longrightarrow} X(\omega)\}\right) = 1,$$

or, equivalently,

$$\mathbb{P}\left(\{\omega \in \Omega : \forall \varepsilon > 0, \exists N > 0 \text{ such that } n \geq N \Longrightarrow |X_n(\omega) - X(\omega)| \leq \varepsilon\}\right) = 1.$$

WARNING. The rank N in the event $\{\omega \in \Omega : \forall \varepsilon > 0, \exists N > 0$ such that $n \geq N \Rightarrow |X_n(\omega) - X(\omega)| \leq \varepsilon\}$ depends not only on ε but also a priori of ω (meaning that it is random).

Convergence in probability. We say that X_n converges in probability to X if for every $\varepsilon > 0$,

$$\mathbb{P}\left(|X_n - X| > \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Convergence in $\mathbb{L}^p$ for $p \geq 1$. This notion is commonly defined when the random variables are **real-valued**. In this case, we say that X_n converges in $\mathbb{L}^p$ to X if $\mathbb{E}\left[|X_n - X|^p\right] \to 0$ as $n \to \infty$.

WARNING. If X_n converges in $\mathbb{L}^1$ to X, then $\mathbb{E}\left[X_n\right] \to \mathbb{E}\left[X\right]$, but the converse is false in general.

2 Links between the different notions of convergence

We have

$$\text{almost sure convergence} \quad \Longrightarrow \quad \text{convergence in probability}$$

and

$$\text{convergence in } \mathbb{L}^p \quad \Longrightarrow \quad \text{convergence in probability}$$

3 In practice, how to show an almost sure convergence?

Strong law of large numbers. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. real valued random variables. Assume that X_1 is integrable. Then

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[X_1\right]$$

almost surely.

By applying Borel-Cantelli lemmas. It is sometimes possible to show almost sure convergence by hand by using Borel–Cantelli. For example, if for every $\varepsilon > 0$ we have

$$\sum_{n\geq 1} \mathbb{P}\left(|X_n - X| > \varepsilon\right) < \infty,$$

then X_n converges almost surely to X.

In practice, to bound quantities of the form $\mathbb{P}\left(|X_n - X| > \varepsilon\right)$ (especially when X is constant), one often uses Markov's inequality $\mathbb{P}\left(|X_n - X| > \varepsilon\right) < \frac{\mathbb{E}[|X_n - X|]}{\varepsilon}$ (first order moment method), Bienaymé-Tchebychev's inequality $\mathbb{P}\left(|X_n - X| > \varepsilon\right) = \mathbb{P}\left((X_n - X)^2 > \varepsilon^2\right) < \frac{\mathbb{E}\left[(X_n - X)^2\right]}{\varepsilon^2}$ (second order moment method) or the fact that for every $\lambda > 0$, $\mathbb{P}\left(|X_n - X| > \varepsilon\right) = \mathbb{P}\left(e^{\lambda|X_n - X|} \geq e^{\lambda\varepsilon}\right) \leq e^{-\lambda\varepsilon}\mathbb{E}\left[e^{\lambda|X_n - X|}\right]$ by choosing a suitable λ (Chernoff's method or large deviations).

By composition. If $X_n \to X$ almost surely and if f is continuous, then $f(X_n) \to f(X)$ almost surely (it is actually possible to relax the assumption "f is continuous" to "f is almost surely continuous at X").

By joint convergence. If $X_n \to X$ almost surely and if $Y_n \to Y$ almost surely, then $(X_n, Y_n) \to (X, Y)$ almost surely.

By taking subsequences. If $X_n \to X$ in probability, then there exists a subsequence of (X_n) converging almost surely to X.

To show that events are almost sure or negligeable, one often uses the following simple result. Let A and B be two events such that $A \subset B$.

- If $\mathbb{P}(A) = 1$, then $\mathbb{P}(B) = 1$.
- If $\mathbb{P}(B) = 0$, then $\mathbb{P}(A) = 0$.

4 In practice, how to show a convergence in probability?

By using an alternative criterion. X_n converges in probability to X if and only if $\mathbb{E}[\min(|X_n - X|, 1)] \to 0$.

By using the subsequence lemma. X_n converges in probability to X if and only if of every subsequence of (X_n) it is possible to re-extract a subsequence converging almost surely to X (that is, for every increasing function $\phi : \mathbb{N}^* \to \mathbb{N}^*$ there exists an increasing function $\psi : \mathbb{N}^* \to \mathbb{N}^*$ such that$X_{\phi(\psi(n))}$ converges almost surely to X).

By composition. If $X_n \to X$ in probability and if f is continuous, then $f(X_n) \to f(X)$ in probability.

By joint convergence. If $X_n \to X$ in probability and if $Y_n \to Y$ in probability, then $(X_n, Y_n) \to (X, Y)$ in probability.

Show a stronger convergence. For instance by showing that $X_n \to X$ a.s. or in $\mathbb{L}^1$ or in $\mathbb{L}^2$ (using for instance Markov's inequality).

5 In practice, how to show a convergence in $\mathbb{L}^1$?

Enhance a convergence in probability to an $\mathbb{L}^1$ convergence (1/2). One often uses the following result (which is an extension of the dominated convergence theorem):

- if X_n converges in probability to X,
- if $(X_n)_{n\geq 1}$ is uniformly integrable (meaning that $\sup_{n\geq 1} \mathbb{E}\left[|X_n| \mathbb{1}_{|X_n|>A}\right] \to 0$ as $A \to \infty$)

then X_n converges to X in $\mathbb{L}^1$.

Remark. The first assumption is satisfied if $X_n \to X$ almost surely. The second assumption is satisfied in the following cases:

- if the random variables (X_n) are *uniformly bounded*, meaning that there exists $a > 0$ such that $\mathbb{P}(|X_n| \leq a) = 1$ pour tout $n \geq 1$;
- if there exists a nonnegative integrable random variable Z, independent of n such that $|X_n| \leq Z$,
- if there exists $\varepsilon > 0$ such that $\sup_{n\geq 1} \mathbb{E}\left[|X_n|^{1+\varepsilon}\right] < \infty$.

Enhance an almost sure convergence to an $\mathbb{L}^1$ convergence (2/2). For nonnegative random variables, Scheffé's lemma (Exercise sheet 7, exercise 2) states:

- if X_n converges almost surely to X,
- if $X_n \geq 0$ for every $n \geq 1$,
- if $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

Then X_n converges to X in $\mathbb{L}^1$.