Problems and suggested solution

Question 1

[10 Points] Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that for every $n \geq 1$ we have $\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}$ and $\mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}$. For $n \geq 1$ we define $S_n = X_1 + \cdots + X_n$.

(1) **[1 Point]** Show that $\mathbb{E}[X_n] = 0$ for every $n \geq 1$.

(2) **[3 Points]** State the Borel-Cantelli lemmas.

(3) **[5 Points]** Show that almost surely

$$\frac{S_n}{n} \underset{n\to\infty}{\longrightarrow} -1.$$

(4) **[1 Point]** Why is it not possible to apply the strong law of large numbers? Justify your answer.

Solution:

(1) We have

$$\mathbb{E}[X_n] = (n^2-1)\mathbb{P}\left(X_n = n^2-1\right) - 1 \cdot \mathbb{P}(X_n = -1) = (n^2-1)\cdot\frac{1}{n^2} - 1\left(1 - \frac{1}{n^2}\right) = 0.$$

(2) Let $(A_n)_{n\geq 1}$ be a sequence of events.

Borel-Cantelli 1. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$.

Borel-Cantelli 2. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and if $(A_n)_{n\geq 1}$ are independent, then $\mathbb{P}(\limsup A_n) = 1$.

(3) Set $A_n = \{X_n = n^2 - 1\}$. Then since $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, by Borel-Cantelli 1. we have $\mathbb{P}(\limsup A_n) = 0$. As a consequence, almost surely A_n happens finitely often. Thus almost surely there exists $N \geq 1$ such that $n \geq N$ implies $X_n = -1$. Thus, almosty surely, for $n \geq N$:

$$\frac{S_n}{n} = \frac{S_N}{n} - \frac{n-N}{n},$$

which tends to -1 as $n \to \infty$.

(4) The random variables $(X_i)_{i\geq 1}$ do not have the same law, so the strong law of large numbers cannot be applied.

Question 2

[**5 Points**] Let $(E, \mathcal{A})$ and $(F, \mathcal{B})$ be two sets equipped with σ-fields. Recall that on $E \times F$, the product σ-field is defined by $\mathcal{A} \otimes \mathcal{B} = \sigma(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\})$. For $C \in \mathcal{A} \otimes \mathcal{B}$ and $x \in E$, we set

$$C_x = \{y \in F : (x, y) \in C\}.$$

(1) [**3 Points**] Fix $x \in E$. Show that $\mathcal{U} = \{C \in \mathcal{A} \otimes \mathcal{B} : C_x \in \mathcal{B}\}$ is a σ-field on $E \times F$.

(2) [**2 Points**] Show that for every $C \in \mathcal{A} \otimes \mathcal{B}$ and $x \in E$ we have $C_x \in \mathcal{B}$.

Solution:

(1) We check the three items of the definition of a σ-field:

- $E \times F \in \mathcal{U}$ since $(E \times F)_x = F \in \mathcal{B}$.
- If $C \in \mathcal{U}$, then $(C^c)_x = \{y \in F : (x, y) \notin C\} = (C_x)^c \in \mathcal{B}$ because $\mathcal{B}$ is stable by complementation.
- If $(C_i)_{i \geq 1}$ is a sequence of elements of $\mathcal{U}$, then

$$\left(\bigcup_{i \geq 1} C_i \right)_x = \left\{ y \in F : (x, y) \in \bigcup_{i \geq 1} C_i \right\} = \bigcup_{i \geq 1} (C_i)_x \in \mathcal{B}$$

  because $\mathcal{B}$ is stable by countable unions.

(2) Fix $x \in E$. Observe that $\mathcal{U}$ contains all elements of the form $A \times B$ with $A \in \mathcal{A}, B \in \mathcal{B}$. Indeed, $(A \times B)_x = B$ if $x \in A$ and $(A \times B)_x = \varnothing$ if $x \notin A$. As a consequence, since $\mathcal{U}$ is a σ-field by (1), $\mathcal{U}$ contains $\sigma(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\})$, which is precisely $\mathcal{A} \otimes \mathcal{B}$. This implies that for every $C \in \mathcal{A} \otimes \mathcal{B}$ we have $C_x \in \mathcal{B}$.

Question 3

[20 Points] Let $\lambda > 0$ and let X be a real-valued random variable such that $\mathbb{P}(X \geq a) = a^{-\lambda}$ for all $a \geq 1$. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables all having the same law as X. We define for every $n \geq 1$

$$T_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n}.$$

Remark: *In the following, Part 1 and Part 2 can be treated independently: question (6) can be solved without using the other questions.*

Part 1.

(1) **[2 Points]** Show that X has a density and give its expression.

(2) **[4 Points]** As $n \to \infty$, does T_n converge almost surely? Justify your answer.

(3) **[1 Point]** As $n \to \infty$, does T_n converge in probability? Justify your answer.

(4) **[4 Points]** Does $\mathbb{E}[T_n^2]$ converge as $n \to \infty$? Justify your answer.

(5) **[3 Points]** As $n \to \infty$, does T_n converge in L^1 ? Justify your answer.

Part 2.

(6) **[6 Points]** Show that $\frac{\max(X_1,\ldots,X_n)}{n^{1/\lambda}}$ converges in distribution as $n \to \infty$.

Solution:

(1) Observe that the cumulative distribution function of X (cdf) of X is given by $\mathbb{P}(X \leq a) = 1 - a^{-\lambda}$ for $a \geq 1$ and $\mathbb{P}(X \leq a) = 0$ for $a < 1$. The cdf is piecewise C^1, so X has a density given by $-\mathbb{1}_{x\geq 1}\frac{\mathrm{d}}{\mathrm{d}x}x^{-\lambda} = \mathbb{1}_{x\geq 1}\frac{\lambda}{x^{\lambda+1}}$

(2) Yes, T_n converges almost surely. Observe that $\mathbb{P}(X \geq 1) = 1$, and that $\mathbb{P}(\ln(X) \geq a) = e^{-\lambda a}$ for every $a \geq 0$. Thus $\ln(X)$ follows an exponential law of parameter λ. In addition,

$$\ln(T_n) = \frac{1}{n}\sum_{i=1}^{n}\ln(X_i).$$

By the composition principle, the random variables $\ln(X_1), \ldots, \ln(X_n)$ are independent with same law distributed as an exponential random variable of parameter λ. By the strong law of large numbers, $\ln(T_n)$ converges almost surely to $1/\lambda$. By continuity of the exponential function, it follows that T_n converges almost surely to $\exp(1/\lambda)$.

(3) Yes, T_n converges in probability to $\exp(1/\lambda)$: we saw in the lecture that almost sure convergence implies convergence in probability.

(4) Write

$$\mathbb{E}\left[T_n^2\right] = \mathbb{E}\left[\prod_{i=1}^{n} X_i^{2/n}\right] = \prod_{i=1}^{n}\mathbb{E}\left[X_i^{2/n}\right] = \mathbb{E}\left[X^{2/n}\right]^n,$$

where we have used the independence of $(X_i)_{1\leq i\leq n}$ for the second equality and the fact that these random variables all have the same law for the last equality. To compute $\mathbb{E}\left[X^{2/n}\right]$ using (1) and the transfer theorem:

$$\mathbb{E}\left[X^{2/n}\right] = \int_1^\infty x^{2/n} \cdot \frac{\lambda}{x^{\lambda+1}}\mathrm{d}x = \int_1^\infty \frac{\lambda}{x^{\lambda-2/n+1}}\mathrm{d}x$$

which is finite for n such that $\lambda - 2/n > 0$. Thus for n sufficiently large $\mathbb{E}\left[X^{2/n}\right] < \infty$ and

$$\mathbb{E}\left[X^{2/n}\right] = \frac{\lambda n}{\lambda n - 2} = 1 + \frac{2}{\lambda n - 2}.$$

Thus, using the Taylor expansion $\ln(1+x) = x + o(x)$ as $x \to 0$:

$$\begin{aligned}\mathbb{E}\left[T_n^2\right] = \left(1+\frac{2}{\lambda n-2}\right)^n = \exp\left(n\ln\left(1+\frac{2}{\lambda n-2}\right)\right) &= \exp\left(n\left(\frac{2}{\lambda n-2}+o\left(\frac{1}{n}\right)\right)\right)\\ &= \exp\left(\frac{2}{\lambda}+o(1)\right)\end{aligned}$$

which converges to $\exp(2/\lambda)$ so the answer of the question is yes.

(5) The answer is yes.

Solution 1. We check that $(T_n)_{n\geq 1}$ converges in probability and is uniformly integrable. The convergence in probability has been established in (2) and uniform integrability comes from the fact that (T_n) is bounded in L^2 since $\mathbb{E}\left[T_n^2\right]$ converges as $n \to \infty$ (we saw in the lecture that a sequence of random variables bounded in L^p for $p > 1$ is uniformly integrable).

For Solutions 2 and 3, we first show that $\mathbb{E}\left[T_n\right] \to \exp(1/\lambda)$. As in question (4), we have

$$\mathbb{E}\left[T_n\right] = \mathbb{E}\left[\prod_{i=1}^n X_i^{1/n}\right] = \prod_{i=1}^n \mathbb{E}\left[X_i^{1/n}\right] = \mathbb{E}\left[X^{1/n}\right]^n,$$

and we similarly compute $\mathbb{E}\left[X^{1/n}\right]$:

$$\mathbb{E}\left[X^{1/n}\right] = \int_1^\infty x^{1/n} \cdot \frac{\lambda}{x^{\lambda+1}}\mathrm{d}x = \frac{\lambda n}{\lambda n - 1} = 1 + \frac{1}{\lambda n - 1}.$$

Thus, similarly to (4):

$$\begin{aligned}\mathbb{E}\left[T_n\right] = \left(1+\frac{1}{\lambda n-1}\right)^n = \exp\left(n\ln\left(1+\frac{1}{\lambda n-1}\right)\right) &= \exp\left(n\left(\frac{1}{\lambda n-1}+o\left(\frac{1}{n}\right)\right)\right)\\ &= \exp\left(\frac{1}{\lambda}+o(1)\right).\end{aligned}$$

This entails

$$\mathbb{E}\left[T_n\right] \underset{n\to\infty}{\longrightarrow} \exp(1/\lambda),$$

Solution 2. We show that $\mathbb{E}\left[(T_n - \exp(1/\lambda)^2\right] \to 0$. Indeed, then by the Cauchy-Schwarz inequality $\mathbb{E}\left[|T_n - \exp(1/\lambda)|\right] \leq \mathbb{E}\left[(T_n - \exp(1/\lambda))^2\right]^{1/2} \to 0$. To this end just write

$$\mathbb{E}\left[(T_n - \exp(1/\lambda)^2\right] = \mathbb{E}\left[T_n^2\right] - 2\exp(1/\lambda)\mathbb{E}\left[T_n\right] + \exp(2/\lambda) \quad \underset{n\to\infty}{\longrightarrow} \quad 0$$

since $\mathbb{E}\left[T_n^2\right] \to \exp(2/\lambda)$ and $\mathbb{E}\left[T_n\right] \to \exp(1/\lambda)$.

Solution 3. We have $T_n \geq 0$, $T_n \to \exp(1/\lambda)$ almost surely and $\mathbb{E}\left[T_n\right] \to \exp(1/\lambda)$. Then Scheffé's lemma (seen in the exercise sheet) implies that $T_n \to \exp(1/\lambda)$ in L^1.

(6) We first compute the point-wise limit of the cdf of $\max(X_1, \ldots, X_n)$. First, for $a \leq 0$ we have $\mathbb{P}\left(\max(X_1, \ldots, X_n)/n^{1/\lambda} \leq a\right) = 0$. Next, for $a > 0$, by independence we have for n sufficiently large so that $an^{1/\lambda} \geq 1$:

$$\begin{aligned}
\mathbb{P}\left(\max(X_1, \ldots, X_n)/n^{1/\lambda} \leq a\right) &= \mathbb{P}\left(X_1 \leq a, \cdots, X_n \leq an^{1/\lambda}\right) \\
&= \mathbb{P}\left(X_1 \leq an^{1/\lambda}\right)^n \\
&= (1 - \mathbb{P}\left(X_1 > an^{1/\lambda}\right))^n \\
&= \left(1 - \frac{1}{(an^{1/\lambda})^\lambda}\right)^n
\end{aligned}$$

because $\mathbb{P}\left(X_1 = an^{1/\lambda}\right) = 0$. Thus

$$\begin{aligned}
\mathbb{P}\left(\max(X_1, \ldots, X_n)/n^{1/\lambda} \leq a\right) &= \left(1 - \frac{1}{a^\lambda n}\right)^n \\
&= \exp\left(n \ln\left(1 - \frac{1}{a^\lambda n}\right)\right) \\
&= \exp\left(n\left(-\frac{1}{a^\lambda n} + o\left(\frac{1}{n}\right)\right)\right)
\end{aligned}$$

so that

$$\mathbb{P}\left(\max(X_1, \ldots, X_n)/n^{1/\lambda} \leq a\right) \quad \underset{n\to\infty}{\longrightarrow} \quad e^{-a^{-\lambda}}.$$

Now observe that $F(a) = e^{-a^{-\lambda}} \mathbb{1}_{a \geq 0}$ is the cdf of a certain random variable X. Indeed, F has limit 0 at $-\infty$, limit 1 at ∞, is continuous and weakly increasing. We conclude that $\frac{\max(X_1,\ldots,X_n)}{n^{1/\lambda}}$ converges in distribution to X.

Question 4

[12 Points] Let $(U_i)_{i\geq 1}$ be a sequence of independent and identically distributed random variables, all following the uniform distribution on $[0,1]$. Fix $x_0 \in (0,1)$. We define by induction a sequence of random variables $(X_n)_{n\geq 0}$ as follows: $X_0 = x_0$, and for $n \geq 0$:

$$X_{n+1} = \mathbb{1}_{U_{n+1}>X_n}\frac{X_n}{2} + \mathbb{1}_{U_{n+1}\leq X_n}\frac{X_n+1}{2}.$$

In other words,

$$X_{n+1} = \begin{cases} \frac{X_n}{2} & \text{if} \quad U_{n+1} > X_n \\ \frac{X_n+1}{2} & \text{if} \quad U_{n+1} \leq X_n. \end{cases}$$

Finally, define $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$ for $n \geq 1$.

In this exercise, you may use without proof the following fact (seen in one of the training exercises): Let X, Y be two real-valued random variables, and $\mathcal{A}$ be a σ-field. Assume that Y is independent of $\mathcal{A}$ and that X is $\mathcal{A}$-measurable. Then for any measurable function $g : \mathbb{R}^2 \to \mathbb{R}^+$, we have

$$\mathbb{E}\left[g(X,Y) \mid \mathcal{A}\right] = h(X) \qquad \text{a.s.,} \qquad \text{where} \qquad h(x) = \mathbb{E}\left[g(x,Y)\right].$$

(1) **[4 Points]** Show that $(X_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

(2) **[2 Points]** Show that $(X_n)_{n\geq 0}$ converges almost surely and in L^1.

(3) **[2 Points]** Show that for every $n \geq 0$ we have $2|X_{n+1} - X_n| \geq \min(X_n, 1 - X_n)$.

(4) **[4 Points]** Denote by X_∞ the almost sure limit of $(X_n)_{n\geq 0}$. Show that X_∞ follows a Bernoulli distribution and find its parameter, justifying your answer.

Solution:

(1) First of all, using the fact that $0 \leq x/2 \leq 1$ and $0 \leq (x+1)/2 \leq 1$ for every $0 \leq x \leq 1$, we readily check by induction that for every $n \geq 0$ we have $0 \leq X_n \leq 1$. As a consequence X_n is bounded and thus integrable.

Next, by induction, we check that X_n is $\mathcal{F}_n$ measurable:

- since $X_0 = x_0$ is constant, it is indeed $\mathcal{F}_0$ measurable.
- assume that X_n is $\mathcal{F}_n$ measurable. Then by definition of X_{n+1}, X_{n+1} is $\sigma(U_{n+1}, X_n)$ measurable as a measurable function of (U_{n+1}, X_n). But both X_n and U_{n+1} are $\mathcal{F}_{n+1}$ measurable, so $\sigma(U_{n+1}, X_n) \subset \mathcal{F}_{n+1}$, which shows that X_{n+1} is $\mathcal{F}_{n+1}$ measurable.

Finally we check that for every $n \geq 0$ we have $\mathbb{E}\left[X_{n+1} \mid F_n\right] = X_n$. To this end, write by linearity of conditional expectation and using the fact that X_n is $\mathcal{F}_n$ measurable:

$$\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = \mathbb{E}\left[\mathbb{1}_{U_{n+1}>X_n} \mid \mathcal{F}_n\right]\frac{X_n}{2} + \mathbb{E}\left[\mathbb{1}_{U_{n+1}\leq X_n} \mid \mathcal{F}_n\right]\frac{X_n+1}{2}.$$

Using the fact given in the statement of the exercise twice, we get

$$\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = (1 - X_n) \cdot \frac{X_n}{2} + X_n \cdot \frac{X_n + 1}{2} = X_n,$$

which completes the proof.

(2) We have seen that $(X_n)_{n\geq 0}$ takes its values in $[0,1]$, so that it is a bounded martingale. In converges therefore almost surely and in L^1.

(3) We have either $X_{n+1} = X_n/2$ or $X_{n+1} = (X_n + 1)/2$, so that $2|X_{n+1} - X_n|$ is equal to either X_n or $1 - X_n$. Thus $2|X_{n+1} - X_n|$ is at least equal to the minimum of these two quantities.

(4) By passing to the limit in (3), we get that almost surely $\min(X_\infty, 1 - X_\infty) \leq 0$. Since $X_\infty \in [0,1]$ we conclude that $\mathbb{P}\left(X_\infty \in \{0,1\}\right) = 1$, so that X_∞ follows a Bernoulli distribution. Its parameter is equal to its mean $\mathbb{E}\left[X_\infty\right]$, which by L^1 convergence is the limit of $\mathbb{E}\left[X_n\right]$. But since (X_n) is a martingale we have $\mathbb{E}\left[X_n\right] = \mathbb{E}\left[X_0\right] = x_0$ for every $n \geq 0$. We conclude that X_∞ is a Bernoulli random variable with parameter x_0.

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