Question 1

[**10 Points**] Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that for every $n \geq 1$ we have $\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}$ and $\mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}$. For $n \geq 1$ we define $S_n = X_1 + \cdots + X_n$.

(1) [**1 Point**] Show that $\mathbb{E}[X_n] = 0$ for every $n \geq 1$.

(2) [**3 Points**] State the Borel-Cantelli lemmas.

(3) [**5 Points**] Show that almost surely

$$\frac{S_n}{n} \underset{n\to\infty}{\longrightarrow} -1.$$

(4) [**1 Point**] Why is it not possible to apply the strong law of large numbers? Justify your answer.

Question 2

[5 Points] Let $(E, \mathcal{A})$ and $(F, \mathcal{B})$ be two sets equipped with σ-fields. Recall that on $E \times F$, the product σ-field is defined by $\mathcal{A} \otimes \mathcal{B} = \sigma(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\})$. For $C \in \mathcal{A} \otimes \mathcal{B}$ and $x \in E$, we set

$$C_x = \{y \in F : (x, y) \in C\}.$$

(1) **[3 Points]** Fix $x \in E$. Show that $\mathcal{U} = \{C \in \mathcal{A} \otimes \mathcal{B} : C_x \in \mathcal{B}\}$ is a σ-field on $E \times F$.

(2) **[2 Points]** Show that for every $C \in \mathcal{A} \otimes \mathcal{B}$ and $x \in E$ we have $C_x \in \mathcal{B}$.

Question 3

[20 Points] Let $\lambda > 0$ and let X be a real-valued random variable such that $\mathbb{P}(X \geq a) = a^{-\lambda}$ for all $a \geq 1$. Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables all having the same law as X. We define for every $n \geq 1$

$$T_n = \left(\prod_{i=1}^{n} X_i \right)^{1/n}.$$

Remark: *In the following, Part 1 and Part 2 can be treated independently: question (6) can be solved without using the other questions.*

Part 1.

(1) **[2 Points]** Show that X has a density and give its expression.

(2) **[4 Points]** As $n \to \infty$, does T_n converge almost surely? Justify your answer.

(3) **[1 Point]** As $n \to \infty$, does T_n converge in probability? Justify your answer.

(4) **[4 Points]** Does $\mathbb{E}[T_n^2]$ converge as $n \to \infty$? Justify your answer.

(5) **[3 Points]** As $n \to \infty$, does T_n converge in L^1 ? Justify your answer.

Part 2.

(6) **[6 Points]** Show that $\frac{\max(X_1,\ldots,X_n)}{n^{1/\lambda}}$ converges in distribution as $n \to \infty$.

Question 4

[12 Points] Let $(U_i)_{i\geq 1}$ be a sequence of independent and identically distributed random variables, all following the uniform distribution on $[0,1]$. Fix $x_0 \in (0,1)$. We define by induction a sequence of random variables $(X_n)_{n\geq 0}$ as follows: $X_0 = x_0$, and for $n \geq 0$:

$$X_{n+1} = \mathbb{1}_{U_{n+1} > X_n} \frac{X_n}{2} + \mathbb{1}_{U_{n+1} \leq X_n} \frac{X_n + 1}{2}.$$

In other words,

$$X_{n+1} = \begin{cases} \frac{X_n}{2} & \text{if} \quad U_{n+1} > X_n \\ \frac{X_n+1}{2} & \text{if} \quad U_{n+1} \leq X_n. \end{cases}$$

Finally, define $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$ for $n \geq 1$.

In this exercise, you may use without proof the following fact (seen in one of the training exercises): Let X, Y be two real-valued random variables, and $\mathcal{A}$ be a σ-field. Assume that Y is independent of $\mathcal{A}$ and that X is $\mathcal{A}$-measurable. Then for any measurable function $g : \mathbb{R}^2 \to \mathbb{R}^+$, we have

$$\mathbb{E}\left[g(X,Y) \mid \mathcal{A}\right] = h(X) \qquad \text{a.s.,} \qquad \text{where} \qquad h(x) = \mathbb{E}\left[g(x,Y)\right].$$

(1) **[4 Points]** Show that $(X_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

(2) **[2 Points]** Show that $(X_n)_{n\geq 0}$ converges almost surely and in L^1.

(3) **[2 Points]** Show that for every $n \geq 0$ we have $2|X_{n+1} - X_n| \geq \min(X_n, 1 - X_n)$.

(4) **[4 Points]** Denote by X_∞ the almost sure limit of $(X_n)_{n\geq 0}$. Show that X_∞ follows a Bernoulli distribution and find its parameter, justifying your answer.

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