Sample PT 2023 ETHZ exam **[Total number of points: 50]**

At any point you can use results proved in the lecture without proof, unless explicitely asked for a proof. If you use a result from the lecture, please reference it appropriately.

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

Intermediate steps may be marked.

Exercise 1. [17 points] Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables following the exponential distribution of parameter 1.

(1) [10 points] Let $(A_n)_{n\geq 1}$ be a sequence of events. Give the definition of the event $\limsup A_n$. State and prove the Borel-Cantelli Lemmas.

(2) [2 points] Fix $c > 1$. Show that

$$\mathbb{P}(X_n > c\ln(n) \text{ for infinitely many } n) = 0.$$

(3) [2 points] Fix $c \in (0,1]$ Show that

$$\mathbb{P}(X_n > c\ln(n) \text{ for infinitely many } n) = 1.$$

(4) [3 points] Fix $c > 0$. Compute, with justification, the quantity

$$\mathbb{P}(X_n \leq c\ln(n) \text{ for infinitely many } n).$$

Solution:

(1) Let $(A_n)_{n\geq 1}$ be a sequence of events. We write

$$\limsup A_n = \bigcap_{n\geq 0} \bigcup_{k\geq n} A_k.$$

[1 points] giving the definition of $\limsup A_n$

Borel-Cantelli 1. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$.

[1 point] correct statement

Proof. For $n \geq 1$, $\limsup A_n \subset \bigcup_{k\geq n} A_k$, so by monotonicity

$$\mathbb{P}(\limsup A_n) \leq \mathbb{P}\left(\bigcup_{k=n} A_k\right) \leq \sum_{k=n}^{\infty} \mathbb{P}(A_k) \quad \underset{n\to\infty}{\longrightarrow} \quad 0$$

[1 point] justification of the first inequality

as the remainder of a convergent series.

[1 point] justification of why $\sum_{k=n}^{\infty} \mathbb{P}(A_k) \to 0$

Borel-Cantelli 2. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and if $(A_n)_{n\geq 1}$ are independent, then $\mathbb{P}(\limsup A_n) = 1$.

[1 point] correct statement

Proof. Fix $n \geq \ell \geq 1$ and write

$$\mathbb{P}\left(\bigcap_{k=\ell}^{n} A_k^c\right) = \prod_{k=\ell}^{n} \mathbb{P}\left(A_k^c\right) = \prod_{k=\ell}^{n} (1 - \mathbb{P}(A_k))$$

[1 point] for these equalities with justification

where we have used independence for the first equality. Using the inequality $\ln(1-x) \leq -x$ valid for $0 \leq x \leq 1$, we get

$$\prod_{k=\ell}^{n} (1 - \mathbb{P}(A_k)) = \exp\left(\sum_{k=\ell}^{n} \ln(1 - \mathbb{P}(A_k))\right) \leq \exp\left(-\sum_{k=l}^{n} \mathbb{P}(A_k)\right).$$

[1 point] the last inequality with justification

But $\sum_{k=l}^{n} \mathbb{P}(A_k) \to \infty$ as $n \to \infty$, so

$$\prod_{k=\ell}^{n} (1 - \mathbb{P}(A_k)) \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

[1 point] this convergence → 0 with justification

But

$$\mathbb{P}\left(\bigcap_{k=\ell}^{n} A_k^c\right) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathbb{P}\left(\bigcap_{k=\ell}^{\infty} A_k^c\right)$$

as a decreasing sequence of events. Thus

$$\mathbb{P}\left(\bigcap_{k=\ell}^{\infty} A_k^c\right) = 0.$$

Thus

$$\mathbb{P}\left(\bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} A_k^c\right) = 0$$

[1 point] this probability = 0 with justification

and by complementation we get

$$\mathbb{P}(\limsup A_n) = \mathbb{P}\left(\bigcap_{\ell=1}^{\infty} \bigcup_{k=\ell}^{\infty} A_k\right) = 1 - \mathbb{P}\left(\bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} A_k^c\right) = 1.$$

[1 point] conclusion by complementation

(2) We have $\mathbb{P}(X_n \geq a) = e^{-a}$ for every $a \geq 0$

[1 point] for this expression

, so

$$\mathbb{P}(X_n > c\ln(n)) = e^{-c\ln(n)} = \frac{1}{n^c}.$$

Thus, for $c > 1$, we have $\sum_{n=1}^{\infty} \frac{1}{n^c} < \infty$.

[1 point] for the convergence of this sum

The conclusion follows from the first Borel-Cantelli Lemma, since $\limsup A_n$ is the event "A_n occurs for infinitely many n".

(3) For $c \in (0, 1]$, we have $\sum_{n=1}^{\infty} \frac{1}{n^c} = \infty$.

[1 point] for the divergence of this sum

Since the events $\{X_n > c\ln(n)\}$ are independent, the conclusion follows from the second Borel-Cantelli Lemma.

[1 point] for stating the independence hypothesis

(4) We have $\mathbb{P}(X_n \leq c\ln(n)) = 1 - 1/n^c \to 1$. So $\sum_{n=1}^{\infty} \mathbb{P}(X_n \leq c\ln(n)) = \infty$.

[1 point] for the divergence of this sum

Since the events $\{X_n > c\ln(n)\}$ are independent, by the second Borel-Cantelli Lemma we get

$$\mathbb{P}(X_n \leq c\ln(n) \text{ for infinitely many } n) = 1.$$

[1 point] for the independence hypothesis

[1 point] for using the Borel-Cantelli Lemma

□

Exercise 2. **[14 points]** Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables which follow the uniform distribution on $[0,1]$. Set $Y_n = (X_n)^n$.

(1) **[4 points]** State and prove the transfer theorem.

(2) **[1 point]** Compute (with justification) $\mathbb{E}[X_1]$.

(3) **[2 points]** Let $F : \mathbb{R} \to \mathbb{R}_+$ be measurable. Using the transfer theorem, write $\mathbb{E}[F(Y_n)]$ as an integral on $[0,1]$ with respect to the Lebesgue measure. Please write explicitly with what function and what random variable you apply the transfer theorem with.

(4) **[2 points]** Using the dummy function method, deduce that Y_n is a random variable with a density, and give an expression of this density.

(5) **[1 point]** Show that Y_n converges in probability to 0 as $n \to \infty$.

(6) **[1 point]** Show that Y_n converges in L^1 as $n \to \infty$.

(7) **[3 points]** Does Y_n converge almost surely as $n \to \infty$? Justify your answer.

Solution:

(1) **Transfer theorem.** Let $X : \Omega \to E$ be a random variable and $f : E \to \mathbb{R}_+$ a measurable function. Then $\mathbb{E}[f(X)] = \int_E f(x)\mathbb{P}_X(\mathrm{d}x)$ where $\mathbb{P}_X$ is the law of X.

[1 points] correct statement

Proof. Step 1. Take $f = \mathbb{1}_A$ with $A \in \mathcal{E}$. Then $\mathbb{E}[\mathbb{1}_A(X)] = \mathbb{E}[\mathbb{1}_{X\in A}] = \mathbb{P}(X \in A)$ and $\int_E \mathbb{1}_A(x)\mathbb{P}_X(\mathrm{d}x) = \mathbb{P}_X(A) = \mathbb{P}(X \in A)$.

[1 point] checking for indicators

Step 2. By linearity, the result is true for any nonnegative simple function.

[1 point] getting the result for nonnegative simple functions

We then take a sequence (f_n) of simple functions such that $0 \leq f_n \leq f$ and $f_n \uparrow f$. By step 1:

$$\mathbb{E}[f_n(X)] = \int_E f_n(x)\mathbb{P}_X(\mathrm{d}x)$$

and by monotone convergence (twice)

$$\mathbb{E}[f_n(X)] = \int_\Omega f_n(X(\omega))\mathbb{P}(\mathrm{d}\omega) \quad \underset{n\to\infty}{\longrightarrow} \quad \int_\Omega f(X(\omega))\mathbb{P}(\mathrm{d}\omega) = \mathbb{E}[f(X))]$$

and

$$\int_E f_n(x)\mathbb{P}_X(\mathrm{d}x) \quad \underset{n\to\infty}{\longrightarrow} \quad \int_E f(x)\mathbb{P}_X(\mathrm{d}x).$$

[1 point] Conclusion by monotone convergence

(2) Using the transfer theorem with the function $f(x) = x$ for $x \in [0,1]$, we have

$$\mathbb{E}[X_1] = \int_0^1 x \mathrm{d}x = \frac{1}{2}.$$

[1 point] obtaining the result with justification using the transfer theorem

(3) We apply the transfer theorem with the random variable X_n and the function $f(x) = F(x^n)$ to get

$$\mathbb{E}[F(Y_n)] = \mathbb{E}[F(X_n^n)] = \int_0^1 F(x^n) \mathrm{d}x.$$

[1 point] obtaining the result with justification using the transfer theorem

(4) We perfom the change of variables $y = x^n$, which gives $x = y^{1/n}$, so $\mathrm{d}x = \frac{1}{n} y^{1/n-1} \mathrm{d}y$:

$$\mathbb{E}[F(Y_n)] = \int_0^1 F(x^n) \mathrm{d}x = \int_0^1 F(y) \frac{1}{n} y^{1/n-1} \mathrm{d}y.$$

We conclude that Y_n has density $\frac{1}{n} y^{1/n-1}$ on $[0,1]$.

[1 point] attempting a change of variables

[1 point] correct result

(5) For $\varepsilon \in (0,1)$, $\mathbb{P}(|Y_n| \geq \varepsilon) = 1 - \varepsilon^{1/n} \to 0$ when $n \to \infty$, which gives the result.

[1 point]

(6) using the transfer theorem, since $|Y_n| = Y_n$ we get

$$\mathbb{E}[Y_n] = \int_0^1 x \frac{x^{1/n-1}}{n} \mathrm{d}x = \frac{1}{n} \int_0^1 x^{1/n} \mathrm{d}x = \frac{1}{n} \frac{1}{1+1/n} \quad \underset{n \to \infty}{\longrightarrow} \quad 0.$$

Thus Y_n converges in L^1 to 0.

[1 point] for the correct computation

(7) Fix any $\varepsilon \in]0,1[$. Using the Taylor expansion $\exp(x) = 1+x+o(x)$, which implies that $1-\exp(x) \sim -x$ as $x \to 0$, we have

$$\mathbb{P}(Y_n \geq \varepsilon) = 1 - \varepsilon^{1/n} = 1 - \exp\left(\frac{1}{n}\ln(\varepsilon)\right) \underset{n\to\infty}{\sim} -\frac{1}{n}\ln(\varepsilon),$$

so that

$$\sum_{n\geq 1} \mathbb{P}(Y_n \geq \varepsilon) = \infty.$$

Since the events $\{Y_n \geq \varepsilon\}$ are independent, by the second Borel-Cantelli lemma almost sureley $Y_n \geq \varepsilon$ infinitely often.

[1 points] for application of Borel–Cantelli 2 with $\{Y_n \geq \varepsilon\}$

Similarly, $\mathbb{P}(Y_n < \varepsilon) = \exp(\frac{1}{n}\ln(\varepsilon)) \to 1$, so

$$\sum_{n\geq 1} \mathbb{P}(Y_n \leq \varepsilon) = \infty.$$

Since the events $\{Y_n \leq \varepsilon\}$ are independent, by the second Borel-Cantelli lemma almost surely $Y_n \leq \varepsilon$ infinitely often.

[1 point] for application of Borel–Cantelli 2 with $\{Y_n < \varepsilon\}$

Thus, almost surely $Y_n \geq \varepsilon$ infinitely often and $Y_n \leq \varepsilon$ infinitely often. This shows that a.s. (Y_n) diverges.

[1 point] for the conclusion

□

Exercise 3. **[9 points]** Let Ω be a set and let $\mathcal{A}$ be a σ-field on Ω. Let H be a set of functions from Ω to $\mathbb{R}$ which satisfies the following two properties:

- H contains all constant functions and is stable under increasing limits (that is if $f : E \to \mathbb{R}$ is a funtion with $f = \lim f_n$ with $(f_n)_{n\geq 1}$ is a sequence of elements of H such that $f_n(\omega) \leq f_{n+1}(\omega)$ for every $\omega \in \Omega$ and $n \geq 0$, then $f \in H$).
- H is a vector space (that is, if $a, b \in \mathbb{R}$ and $f, g \in H$ then $af + bg \in H$).

(1) **[2 points]** State the Dynkin Lemma.

(2) **[3 points]** Show that $\mathcal{B} = \{A \in \mathcal{A} : \mathbb{1}_A \in H\}$ is a Dynkin system.

(3) **[4 points]** Let $\mathcal{C} \subset \mathcal{A}$ be a generating π-system of $\mathcal{A}$. Assume that $\mathbb{1}_A \in H$ for every $A \in \mathcal{C}$. Show that H contains all $\mathcal{A}$-measurable real-valued functions.

Hint. First use the Dynkin Lemma to show that H contains all functions of the form $\mathbb{1}_A$ with $A \in \mathcal{A}$.

Solution:

(1) **Dynkin Lemma.** Let Ω be a set and let $\mathcal{C} \subset \mathcal{P}(\Omega)$ be a collection of subsets of Ω. Assume that $\mathcal{C}$ is stable by finite intersections. Then the Dynkin system generated by $\mathcal{C}$ is equal to the σ-field generated by $\mathcal{C}$.

[2 points] for the correct statement.

(2) We check the three properties defining a Dynkin system:

(a) $\Omega \in \mathcal{B}$ because the constant function equal to 1 is in H.

(b) If $A \in \mathcal{B}$, then $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$ because 1 and $\mathbb{1}_A$ are in H and H is a vector space.

(c) If $(A_n)_{n\geq 1}$ is a pairwise disjoint sequence of elements of $\mathcal{B}$, set $A = \cup_{n\geq 1} A_n$ and then observe that

$$\mathbb{1}_A = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{1}_{A_k},$$

where the limit is increasing. In addition $\mathbb{1}_{A_k} \in H$ since H is a vector space. Since H is stable under increasing limits, we conclude that $\mathbb{1}_A$ is in H

[4 points] 1 point for (a), 1 point for (b), 2 points for (c)

(3) *Step 1.* By question (2), $\mathcal{B}$ is a Dynkin system containing $\mathcal{C}$. Therefor $\mathcal{B}$ contains the Dynkin system generated by $\mathcal{C}$. Since $\mathcal{C}$ is a π-system, by the Dynkin Lemma the Dynkin system generated by $\mathcal{C}$ is $\sigma(\mathcal{C})$, which is $\mathcal{A}$. We conclude that $\mathcal{B} = \mathcal{A}$, so that H contains all functions of the form $\mathbb{1}_A$

with $A \in \mathcal{A}$.

[1 point] for step 1

Step 2. Since H is a vector space, by linearity H contains all simple $\mathcal{A}$-measurable functions.

[1 point] for step 2

Step 3. H contains all nonnegative $\mathcal{A}$-measurable functions, since every nonnegative $\mathcal{A}$-measurable function is an increasing limit of $\mathcal{A}$-measurable simple functions and H is stable under increasing limits.

[1 point] for step 3

Step 4. H contains all $\mathcal{A}$-measurable functions, since every $\mathcal{A}$-measurable function can be written as a difference of two nonngative $\mathcal{A}$-measurable functions and H is a vector space.

[1 point] for step 4

□

Exercise 4. **10 points** Fix an integer $n \geq 2$ and let $(U_k)_{1 \leq k \leq n}$ be independent random variables, all following the uniform distribution on $[0,1]$. Define

$$M_n = \max\left(\frac{1}{\sqrt{U_1}}, \dots, \frac{1}{\sqrt{U_n}}\right).$$

(1) **[3 points]** Compute the cumulative distribution function of M_n.

(2) **[2 points]** Show that $x^2 \mathbb{P}(M_n \geq x) \to n$ as $x \to \infty$.

(3) **[2 points]** Let X be a nonnegative real-valued random variable. Show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq u) \mathrm{d}u$.
Note. This question is independent of questions (1) and (2).

(4) **[3 points]** For what values of $p > 0$ do we have $\mathbb{E}\left[M_n^p\right] < \infty$?

Note. You may use the results of the previous questions even if you didn't manage to solve them.

Solution:

(1) For $u \in [0,1]$, we have $\frac{1}{\sqrt{u}} \geq 1$. As a consequence for $x < 1$ we have $\mathbb{P}(M_n \leq x) = 0$.

[1 point for the case $x < 1$]

For $x \geq 1$ we have:

$$\mathbb{P}(M_n \leq x) = \mathbb{P}\left(\frac{1}{\sqrt{U_1}} \leq x, \cdots, \frac{1}{\sqrt{U_n}} \leq x\right) = \mathbb{P}\left(U_1 \geq \frac{1}{x^2}\right) \cdots \mathbb{P}\left(U_n \geq \frac{1}{x^2}\right) = \mathbb{P}\left(U_1 \geq \frac{1}{x^2}\right)^n = \left(1 - \frac{1}{x^2}\right)^n.$$

The second equality follows from independence and the third one from the fact that $(U_i)_{1 \leq i \leq n}$ have same law.

[1 point] for the second equality

[1 point] for the the final result

(2) By (1), for $x \geq 1$ we have

$$\mathbb{P}(M_n \geq x) = 1 - \mathbb{P}(M_n < x) = 1 - \left(1 - \frac{1}{x^2}\right)^n.$$

By the Binomial formula,

$$\left(1 - \frac{1}{x^2}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{x^{2k}},$$

so

$$x^2 \mathbb{P}(M_n \geq x) = x^2\left(1 - \left(1 - \frac{1}{x^2}\right)^n\right) = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{x^{2k-2}} = n + \sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{x^{2k-2}} \quad \underset{x \to \infty}{\longrightarrow} \quad n$$

[2 points] for justification and conclusion

(3) By Fubini-Tonelli's theorem for nonnegative functions:

$$\mathbb{E}[X] = \mathbb{E}\left[\int_0^\infty 1_{u \le X} \mathrm{d}u\right] = \int_0^\infty \mathbb{E}[1_{u \le X}] \mathrm{d}u = \int_1^\infty \mathbb{P}(X \ge u) \mathrm{d}u.$$

[1 point] for making $\mathbb{E}$ and $\int$ appear at the same time

[1 point] for citing Fubini theorem for nonnegative functions

(4) Here $M_n \ge 0$.

[1 point] for checking $M_n \ge 0$

We have by question (3)

$$\mathbb{E}\left[M_n^p\right] = \int_0^\infty \mathbb{P}\left(M_n^p \ge u\right) \mathrm{d}u = \int_0^\infty \mathbb{P}\left(M_n \ge u^{1/p}\right) \mathrm{d}u = 1 + \int_1^\infty \mathbb{P}\left(M_n \ge u^{1/p}\right) \mathrm{d}u.$$

[1 point for this expression]

But for $u \ge 1$:

$$\mathbb{P}\left(M_n \ge u^{1/p}\right) = \frac{1}{u^{2/p}}\left(u^{2/p}\mathbb{P}\left(M_n \ge u^{1/p}\right)\right) \underset{u \to \infty}{\sim} \frac{n}{u^{2/p}}.$$

by question (2). But $1/u^{2/p}$ is integrable at infinity if and only if $2/p > 1$ (Riemann integral). Thus $\mathbb{E}\left[M_n^p\right] < \infty$ if and only if $p < 2$.

[1 points] conclusion.

□