Sample PT 2023 ETHZ exam **[Total number of points: 50]**

At any point you can use results proved in the lecture without proof, unless explicitely asked for a proof. If you use a result from the lecture, please reference it appropriately.

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

Intermediate steps may be marked.

Exercise 1. **[17 points]** Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables following the exponential distribution of parameter 1.

(1) **[10 points]** Let $(A_n)_{n\geq 1}$ be a sequence of events. Give the definition of the event $\limsup A_n$. State and prove the Borel-Cantelli Lemmas.

(2) **[2 points]** Fix $c > 1$. Show that

$$\mathbb{P}(X_n > c\ln(n) \text{ for infinitely many } n) = 0.$$

(3) **[2 points]** Fix $c \in (0, 1]$. Show that

$$\mathbb{P}(X_n > c\ln(n) \text{ for infinitely many } n) = 1.$$

(4) **[3 points]** Fix $c > 0$. Compute, with justification, the quantity

$$\mathbb{P}(X_n \leq c\ln(n) \text{ for infinitely many } n).$$

Exercise 2. **[14 points]** Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables which follow the uniform distribution on $[0,1]$. Set $Y_n = (X_n)^n$.

(1) **[4 points]** State and prove the transfer theorem.

(2) **[1 point]** Compute (with justification) $\mathbb{E}[X_1]$.

(3) **[2 points]** Let $F : \mathbb{R} \to \mathbb{R}_+$ be measurable. Using the transfer theorem, write $\mathbb{E}[F(Y_n)]$ as an integral on $[0,1]$ with respect to the Lebesgue measure. Please write explicitly with what function and what random variable you apply the transfer theorem with.

(4) **[2 points]** Using the dummy function method, deduce that Y_n is a random variable with a density, and give an expression of this density.

(5) **[1 point]** Show that Y_n converges in probability to 0 as $n \to \infty$.

(6) **[1 point]** Show that Y_n converges in L^1 as $n \to \infty$.

(7) **[3 points]** Does Y_n converge almost surely as $n \to \infty$? Justify your answer.

Exercise 3. **[9 points]** Let Ω be a set and let $\mathcal{A}$ be a σ-field on Ω. Let H be a set of functions from Ω to $\mathbb{R}$ which satisfies the following two properties:

- H contains all constant functions and is stable under increasing limits (that is if $f : E \to \mathbb{R}$ is a funtion with $f = \lim f_n$ with $(f_n)_{n\geq 1}$ is a sequence of elements of H such that $f_n(\omega) \leq f_{n+1}(\omega)$ for every $\omega \in \Omega$ and $n \geq 0$, then $f \in H$).
- H is a vector space (that is, if $a, b \in \mathbb{R}$ and $f, g \in H$ then $af + bg \in H$).

(1) **[2 points]** State the Dynkin Lemma.

(2) **[3 points]** Show that $\mathcal{B} = \{A \in \mathcal{A} : \mathbb{1}_A \in H\}$ is a Dynkin system.

(3) **[4 points]** Let $\mathcal{C} \subset \mathcal{A}$ be a generating π-system of $\mathcal{A}$. Assume that $\mathbb{1}_A \in H$ for every $A \in \mathcal{C}$. Show that H contains all $\mathcal{A}$-measurable real-valued functions.

Hint. First use the Dynkin Lemma to show that H contains all functions of the form $\mathbb{1}_A$ with $A \in \mathcal{A}$.

Exercise 4. **10 points** Fix an integer $n \geq 2$ and let $(U_k)_{1 \leq k \leq n}$ be independent random variables, all following the uniform distribution on $[0,1]$. Define

$$M_n = \max\left(\frac{1}{\sqrt{U_1}}, \dots, \frac{1}{\sqrt{U_n}}\right).$$

(1) **[3 points]** Compute the cumulative distribution function of M_n.

(2) **[2 points]** Show that $x^2 \mathbb{P}(M_n \geq x) \to n$ as $x \to \infty$.

(3) **[2 points]** Let X be a nonnegative real-valued random variable. Show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq u)\,\mathrm{d}u$.
Note. This question is independent of questions (1) and (2).

(4) **[3 points]** For what values of $p > 0$ do we have $\mathbb{E}\left[M_n^p\right] < \infty$?

Note. You may use the results of the previous questions even if you didn't manage to solve them.