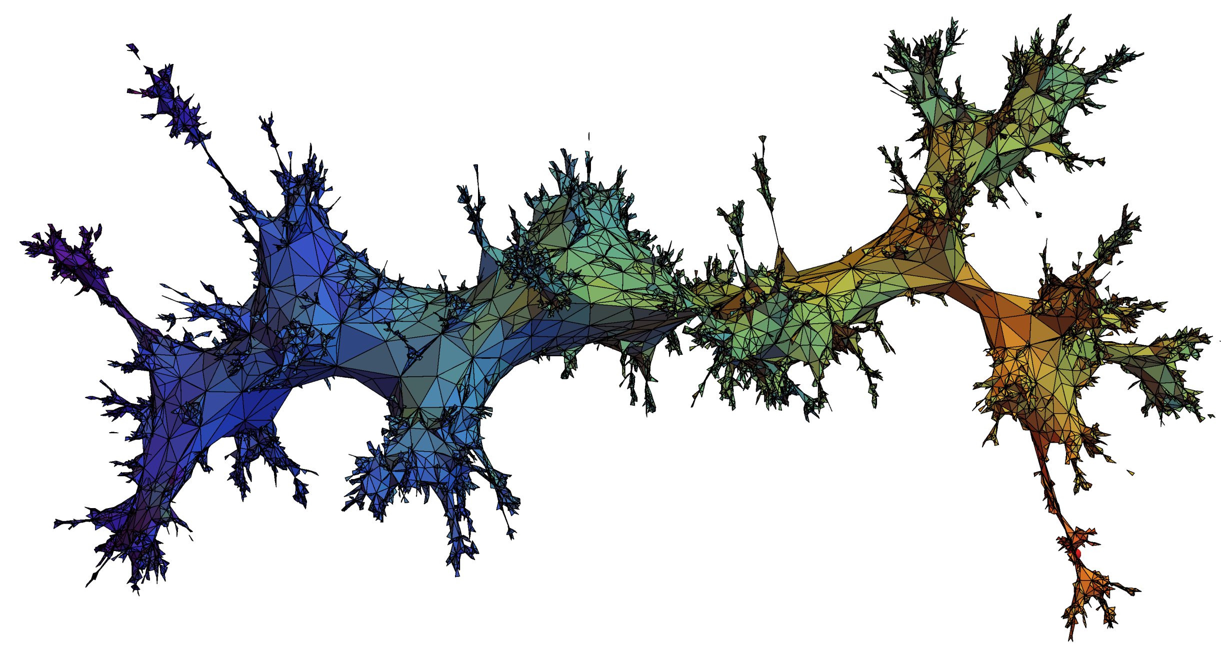
The Brownian Universe

Limacodidae Caterpillar © John Horstman



Brownian sphere © Igor Kortchemski

Igor Kortchemski

ETH Zürich

Probability Theory - Autumn 2023

General framework

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To answer this question, a possibility is to find a continuous object X such that $X_n \to X$ as $n \to \infty$.

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- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging to X, then X_n and Y_n "roughly" have the same properties for n large.

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- *In what space do the objects live?* Here, a metric space (E, d).
- *What is the sense of this convergence when these objects are random?* Here, convergence in distribution:

$$\mathbb{E}\left[F(X_n)\right] \quad \underset{n \to \infty}{\longrightarrow} \quad \mathbb{E}\left[F(X)\right]$$

for every continuous bounded function $F : E \to \mathbb{R}$.

Outline

I. RANDOM PATHS (1951)

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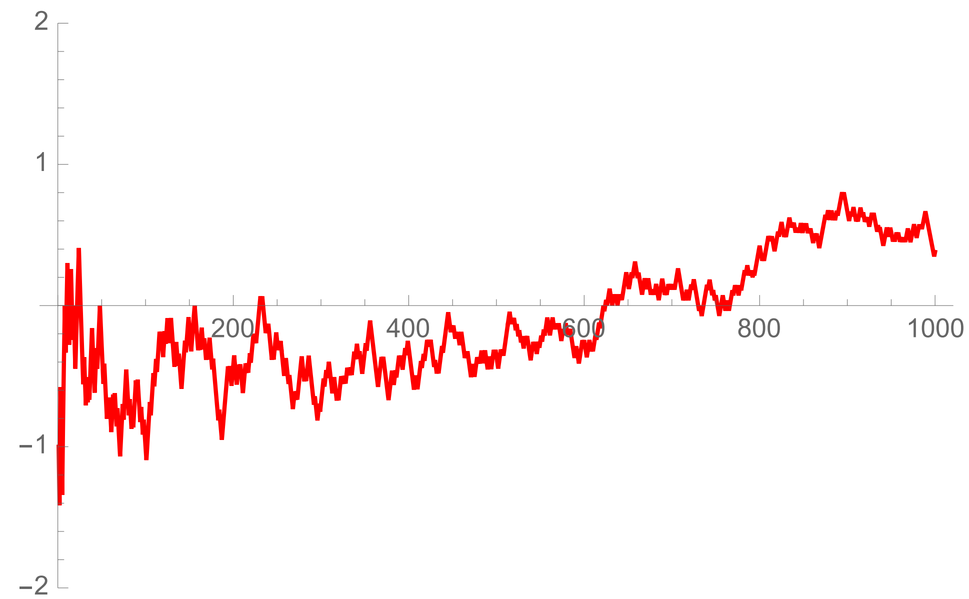

Figure: A simulation of $\left(\frac{S_k}{\sqrt{k}}\right)_{1 \leqslant k \leqslant n}$ for $n = 1000$.

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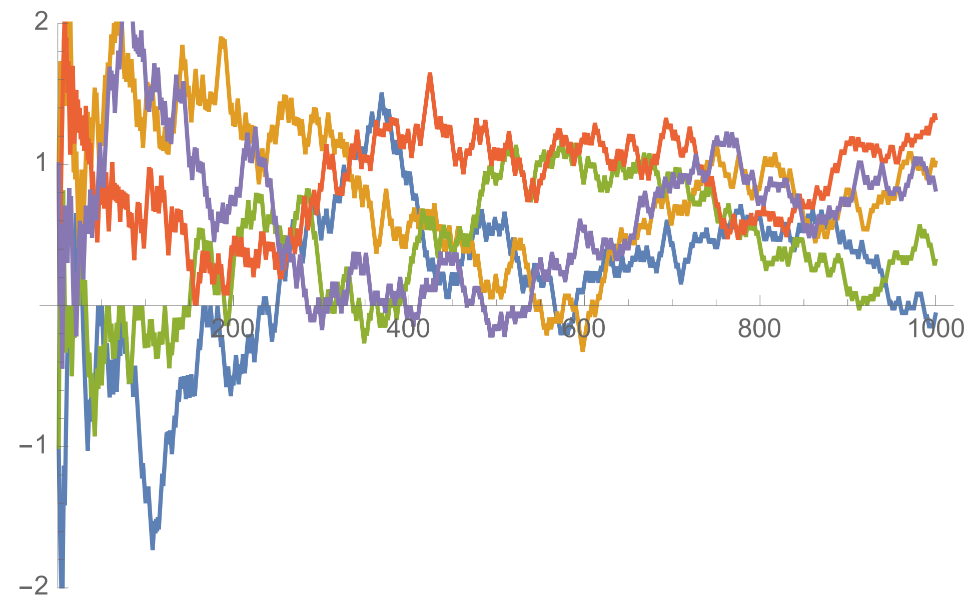

Figure: Five simulations of $\left(\frac{S_k}{\sqrt{k}}\right)_{1 \leqslant k \leqslant n}$ for $n = 1000$.

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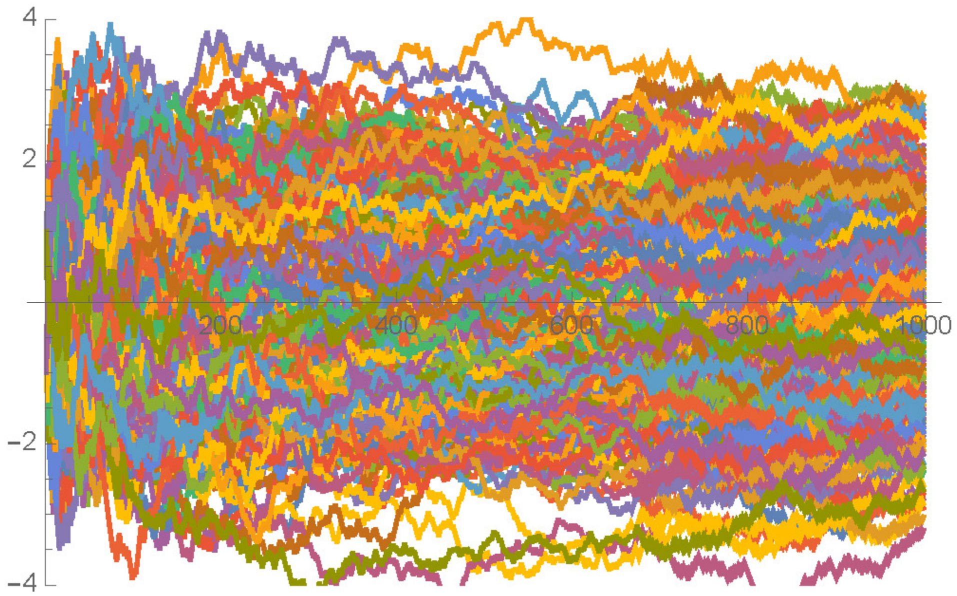

Figure: 1000 simulations of $\left(\frac{S_k}{\sqrt{k}}\right)_{1 \leqslant k \leqslant n}$ for $n = 1000$.

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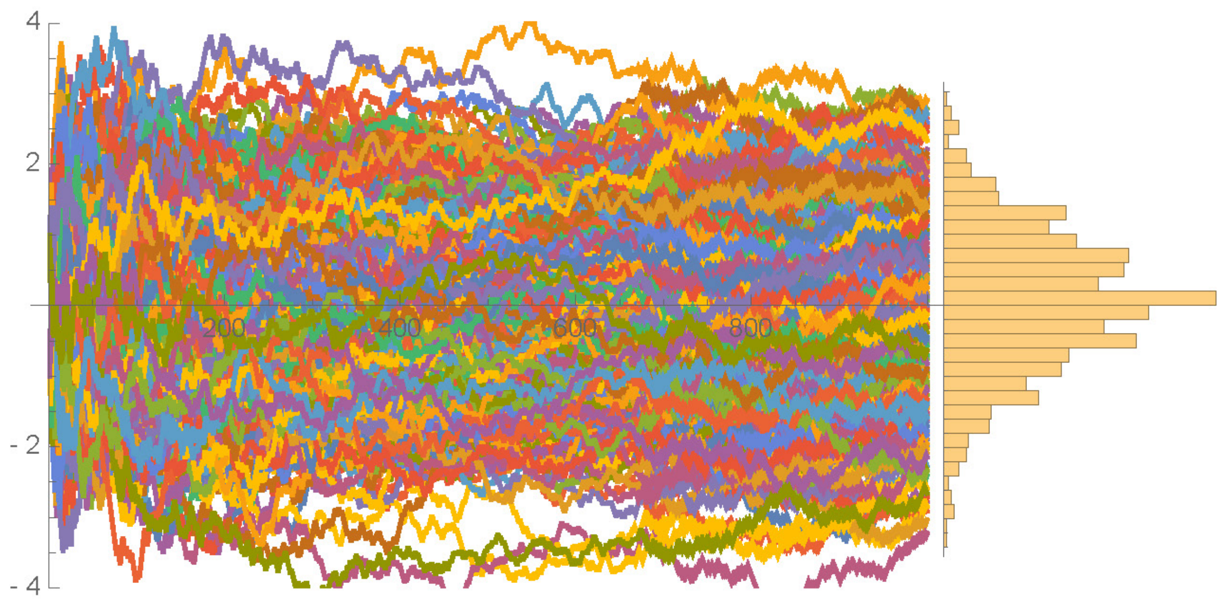

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Consequence: for every $a<b$,

$$\mathbb{P}\left(a<\frac{S_n}{\sigma\sqrt{n}}<b\right) \quad \xrightarrow[n\to\infty]{} \quad \int_a^b dx\, \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

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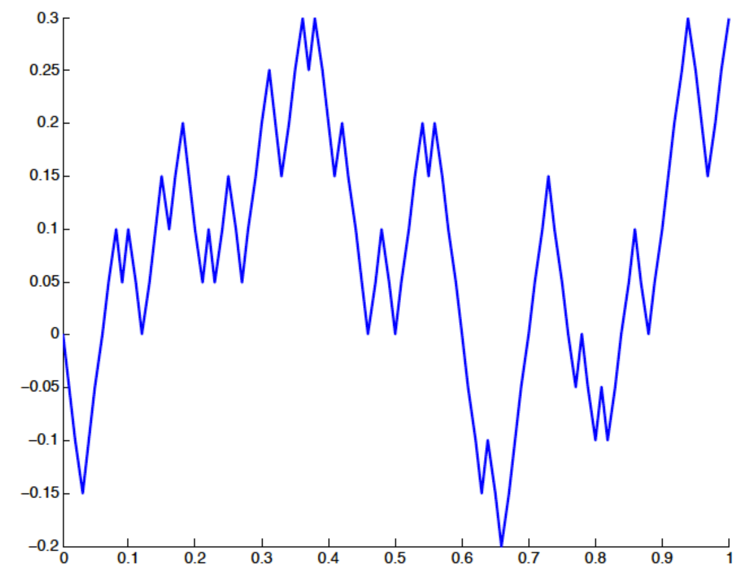

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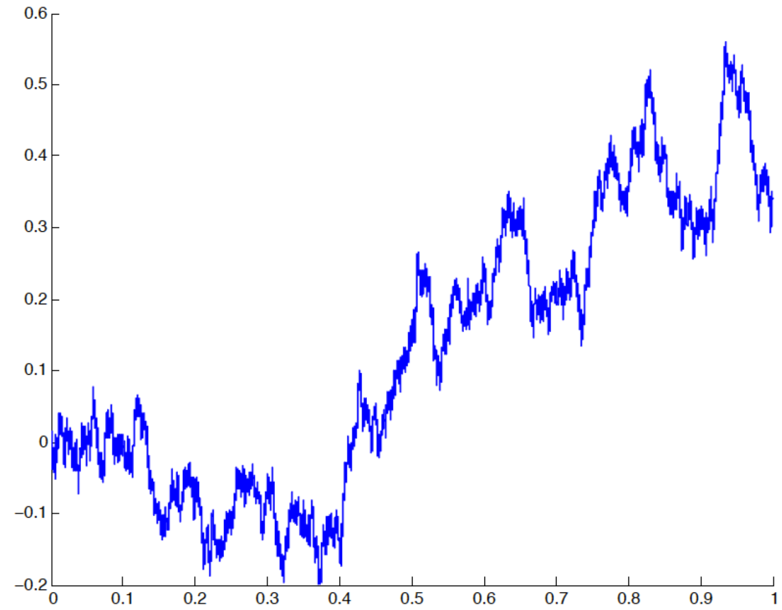

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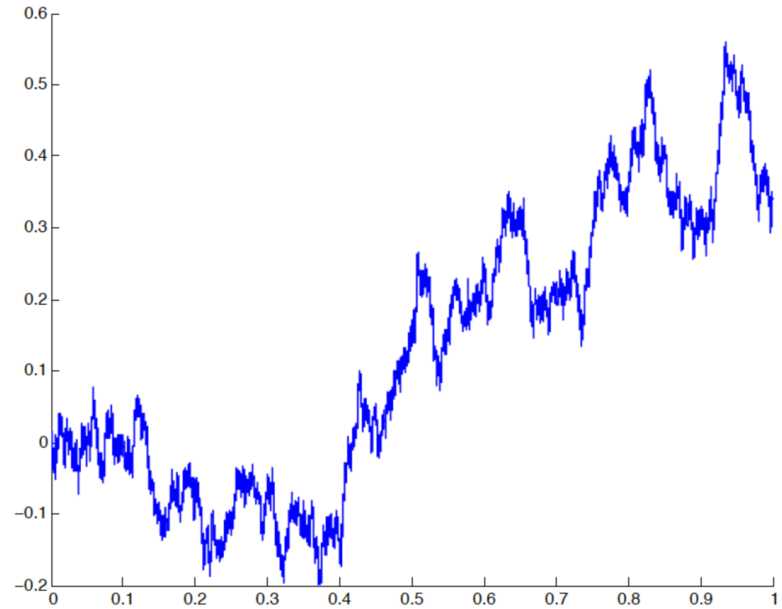

Here, $E = \mathcal{C}([0,1], \mathbb{R})$ is the space of real-valued continuous functions on $[0,1]$ equipped with the uniform norm.

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Theorem (Donsker, 1951)

We have the convergence in distribution

$$\left(\frac{S_{nt}}{\sigma\sqrt{n}}, t \geqslant 0\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (W_t, t \geqslant 0),$$

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$$\mathbb{P}\left(\frac{\max_{0 \leqslant i \leqslant n} S_i}{\sigma\sqrt{n}} > a\right) \quad \xrightarrow[n\to\infty]{} \quad \mathbb{P}\left(\sup_{0 \leqslant t \leqslant 1} W_t > a\right)$$

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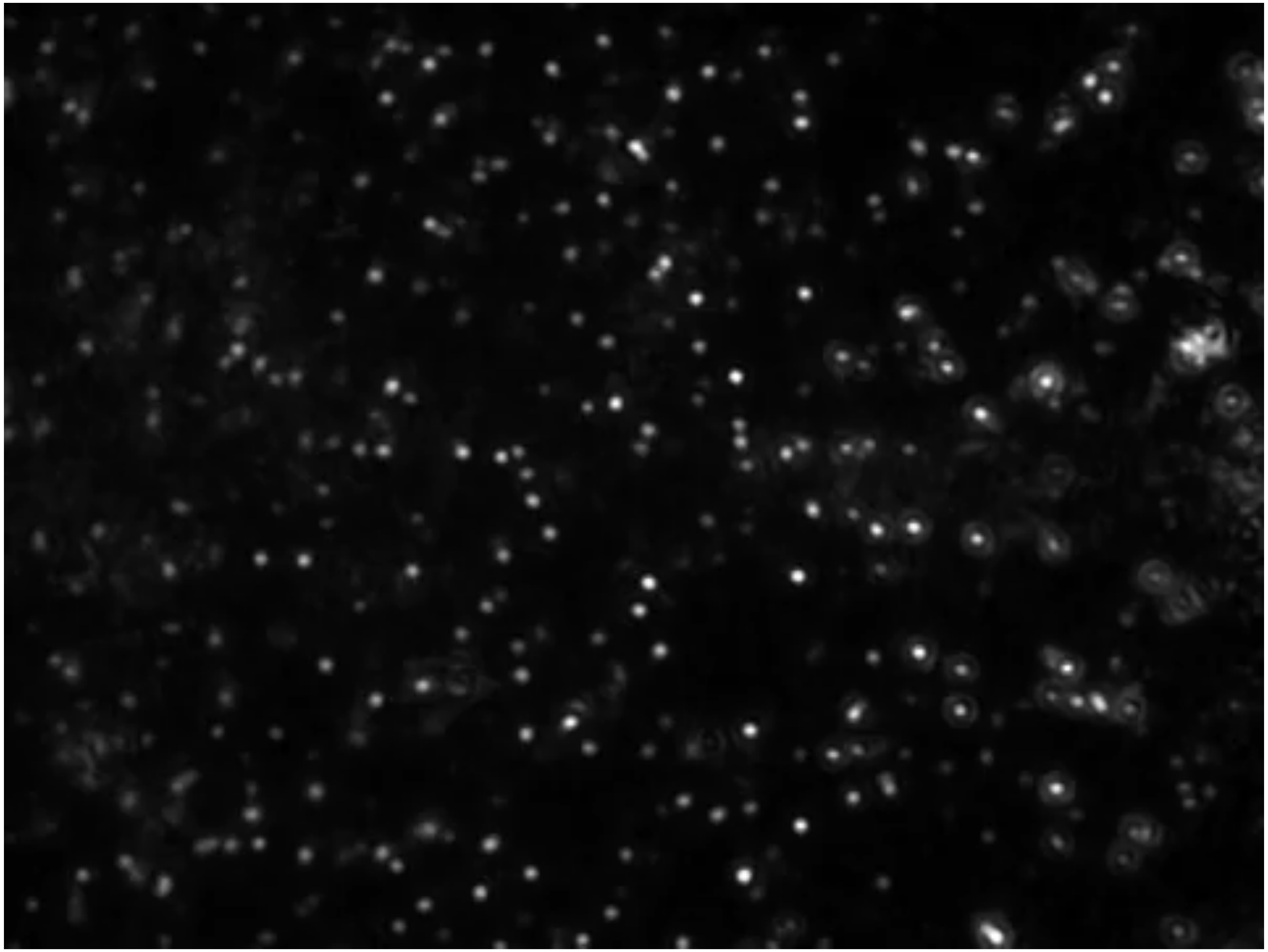

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"This is a case where it's really natural to think of those continuous functions without derivatives that mathematicians have imagined, and which were wrongly regarded as mere mathematical curiosities, since experience can suggest them."

– Jean Perrin

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- 1923: Wiener gives a mathematical construction of Brownian motion.

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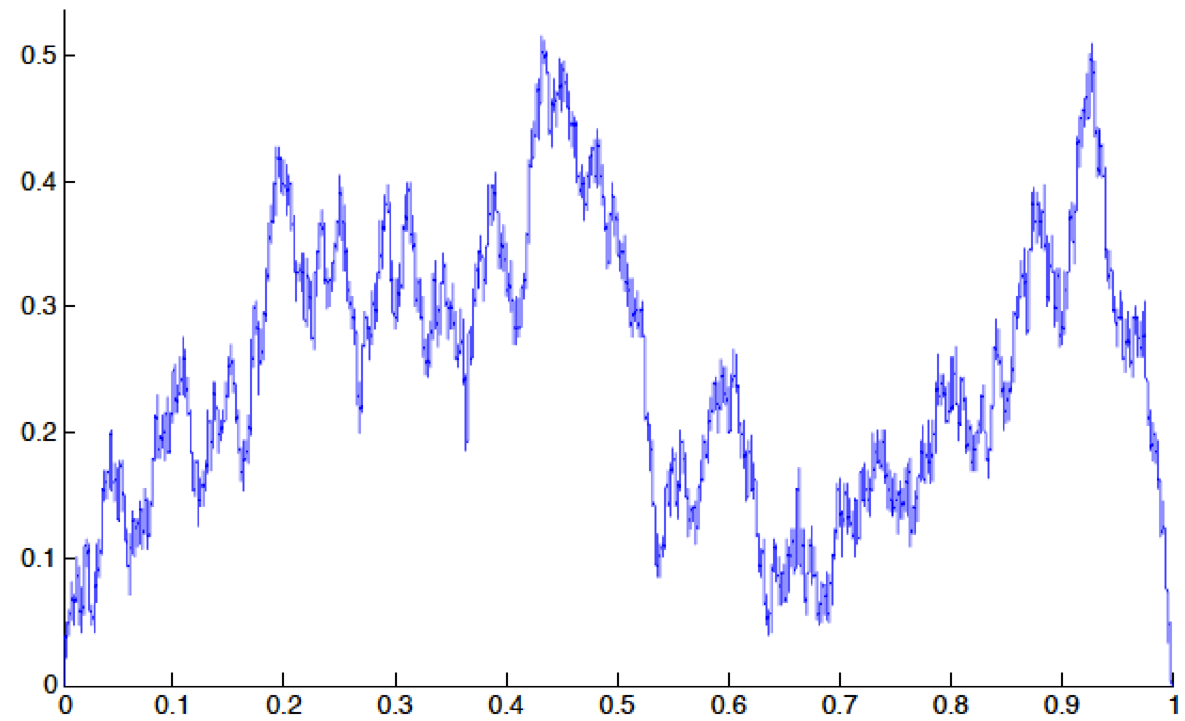

Theorem (conditioned Donsker, Kaigh '75)

Let $(X_n)_{n\geqslant 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\sigma^2 = \mathbb{E}\left[{X_1}^2\right] < \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then:

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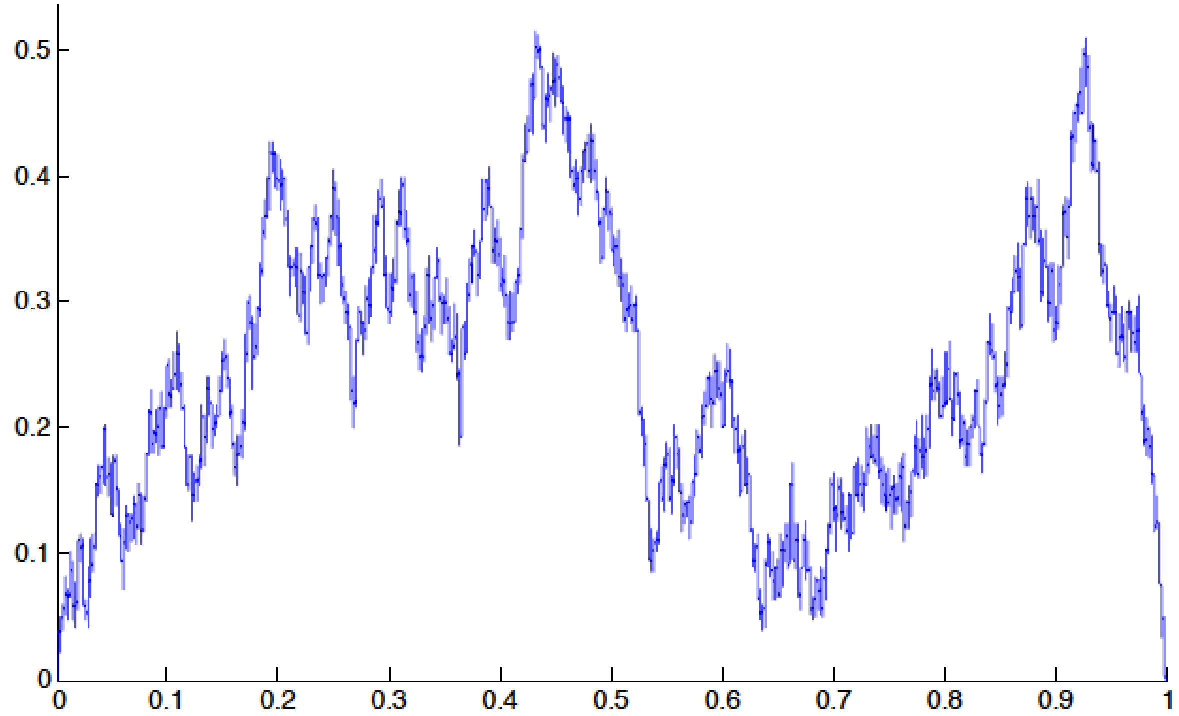

The Brownian excursion can be seen as Brownian motion $(W_t, 0 \leqslant t \leqslant 1)$ conditioned by $W_1 = 0$ and $W_t > 0$ for $t \in (0, 1)$.

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I. Random paths (1951)

II. Random trees (1994)

III. Random surfaces (2004)

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Motivations:

⇝ Computer Science: data structures, analysis of algorithms, networks, etc.

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- Probability: trees are building blocks of several models of random graphs, having rich probabilistic properties.

Plane trees

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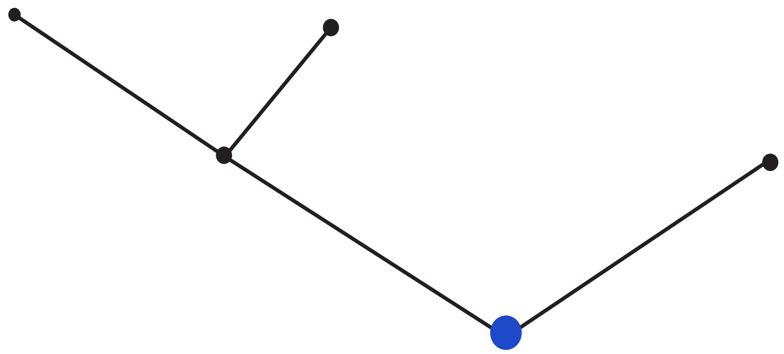

Figure: Two different plane trees

Plane trees

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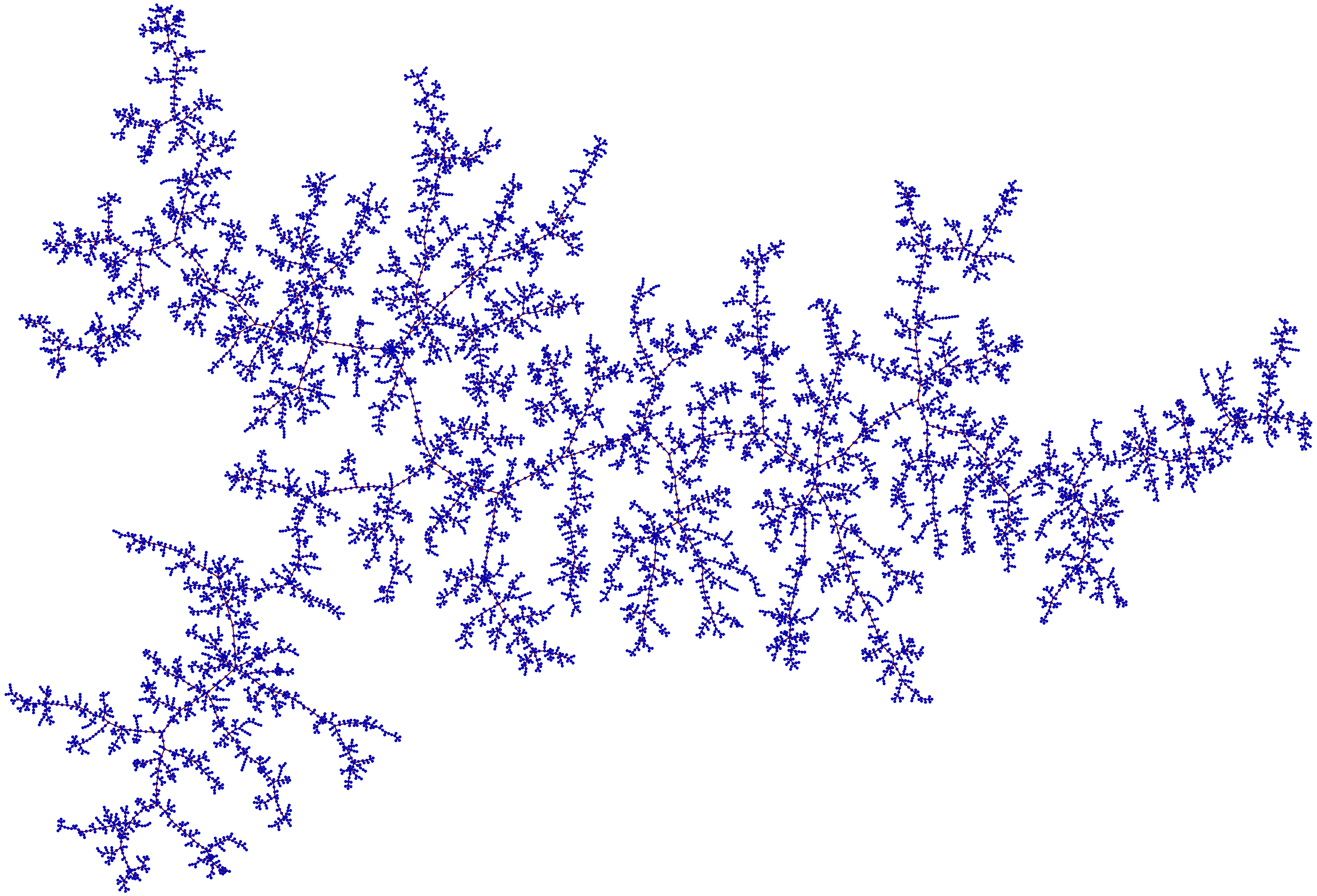
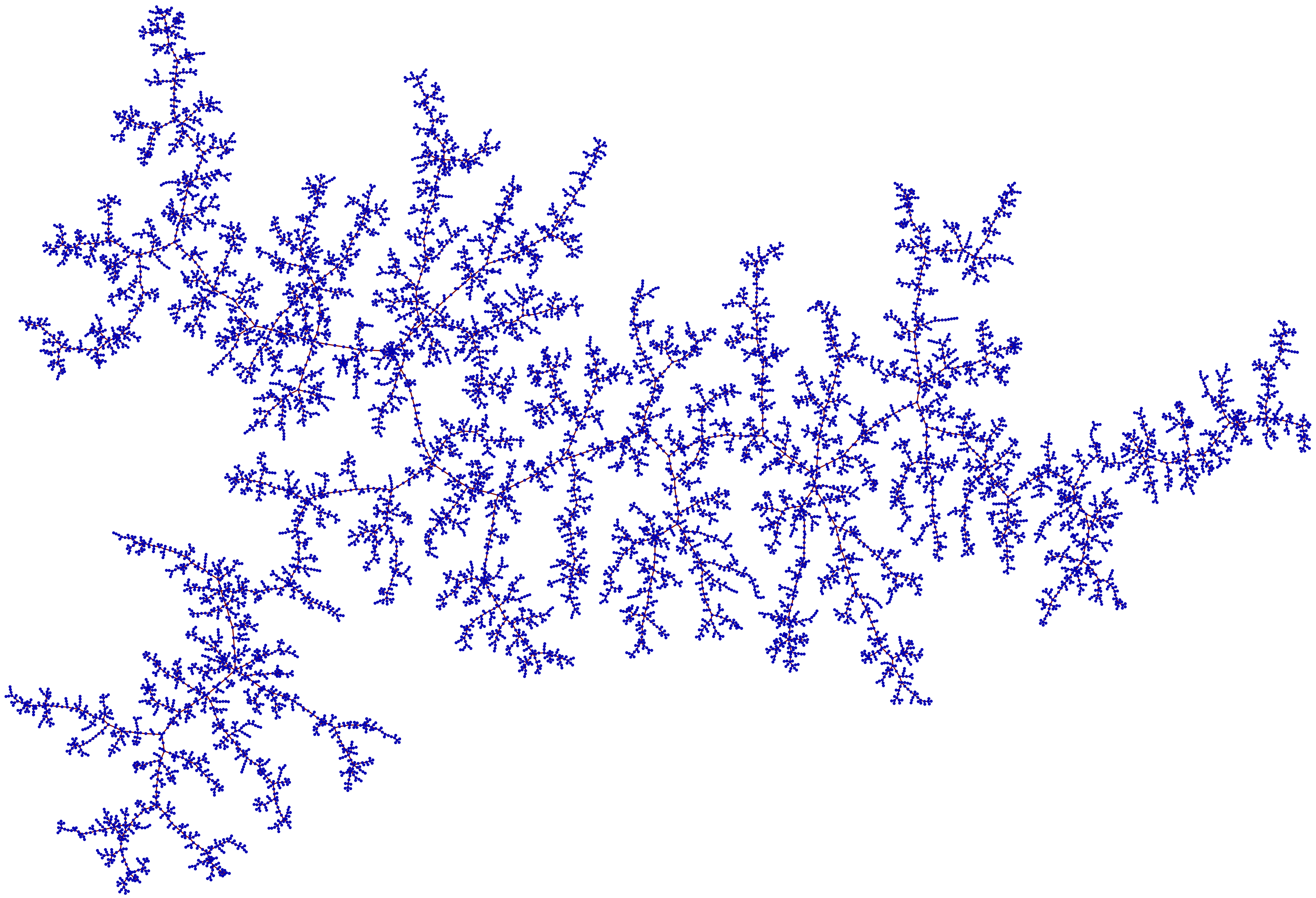
Figure: Two different plane trees

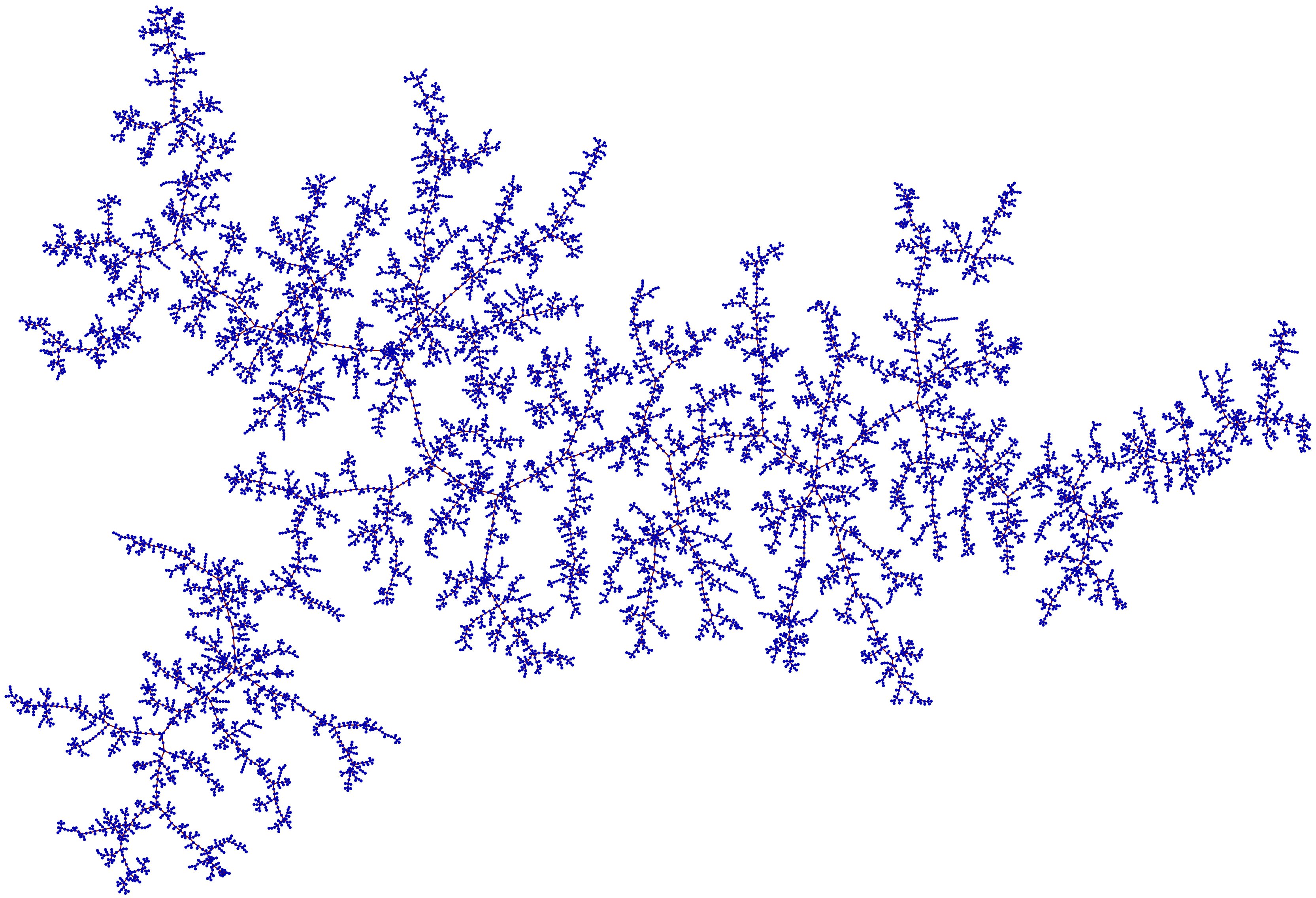
$\leadsto$ Question: What does a large typical plane tree look like?





Let $\mathcal{T}_n$ be a uniform plane tree with n vertices chosen uniformly at random.

⇝ What is the order of magnitude of the diameter of $\mathcal{T}_n$?



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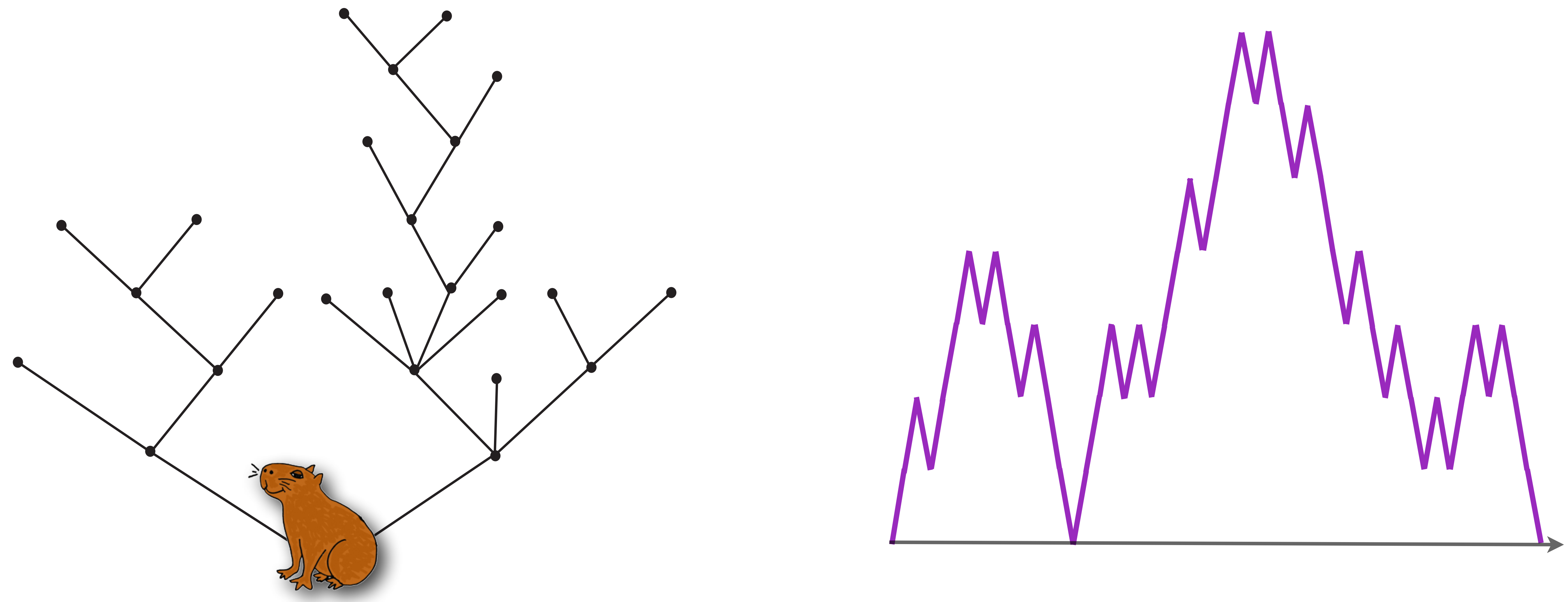
⇝ wooclap.com ; code **probability**.

What metric space for $\mathcal{T}_n$?

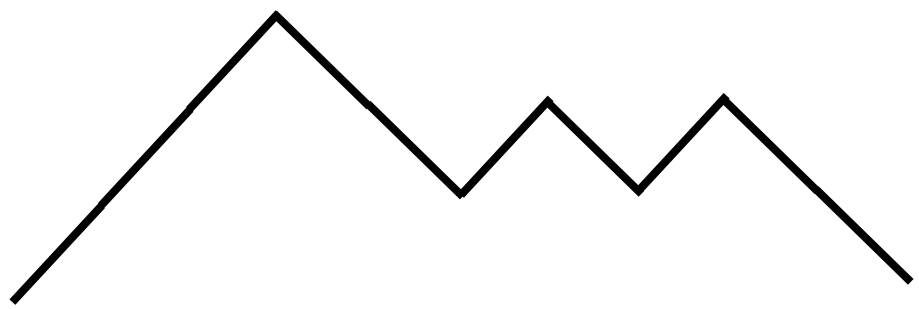
Coding a tree by its contour function

We code a tree τ by its contour function $C(\tau)$:



Coding a tree by its contour function

Knowing the contour function, it is easy to recover the tree:



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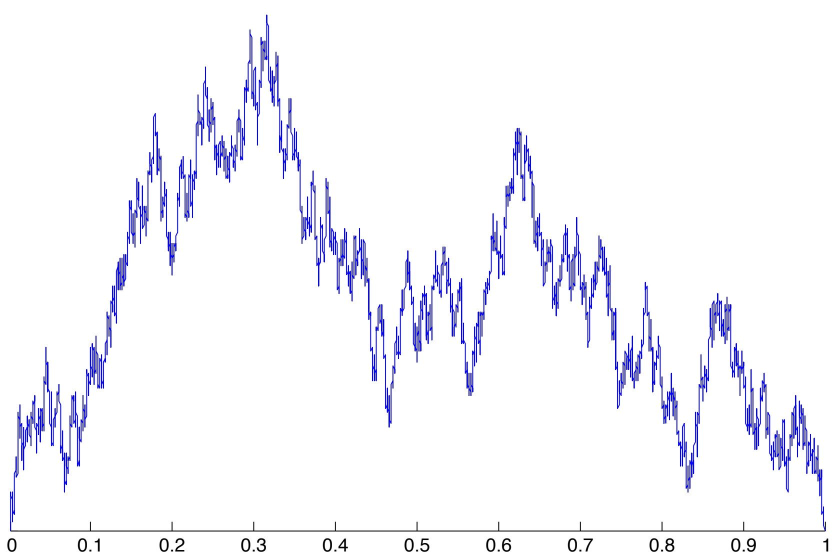

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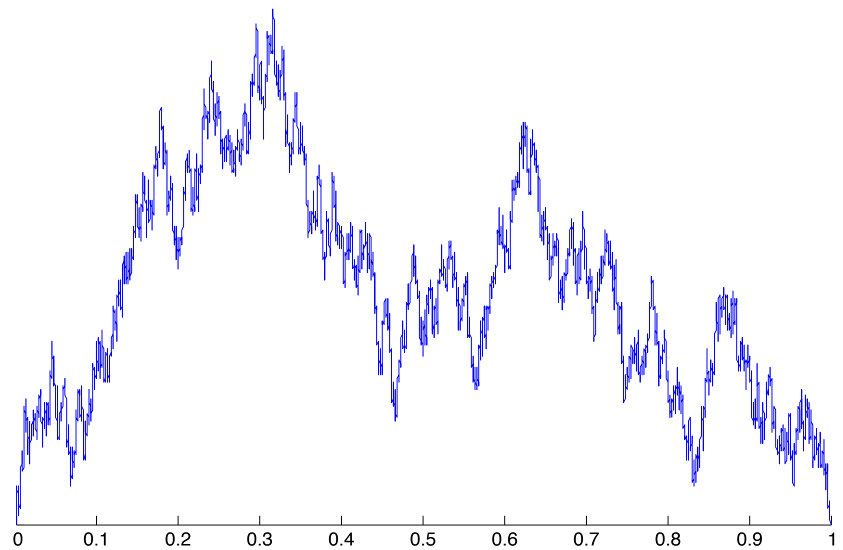

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$$(C_0, \ldots, C_{2(n-1)}) \overset{(d)}{=} (S_0, \ldots, S_{2(n-1)}) \quad \text{under } \mathbb{P}(\cdots | S_{2n-2} = 0, S_i \geqslant 0 \text{ for } i < 2n-2)$$

where $(S_k)_{k \geqslant 0}$ is the random walk with jumps ± 1 with probability $1/2$.

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⇝ We get the desired result with the extension of Donsker's theorem to the conditioned case.

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⇝ Extension to a more general class of random plane trees: Bienaymé–Galton–Watson trees with critical finite variance offspring distribution, conditioned on having a large number of vertices.

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⇝ Consequence 2 : Yes, when we view $\mathcal{T}_n$ as a compact metric space by equipping its vertices with the graph distance.

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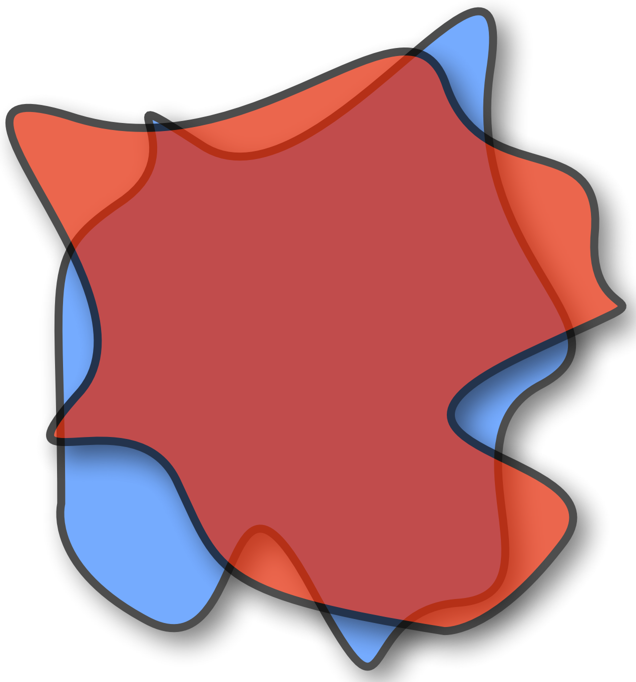

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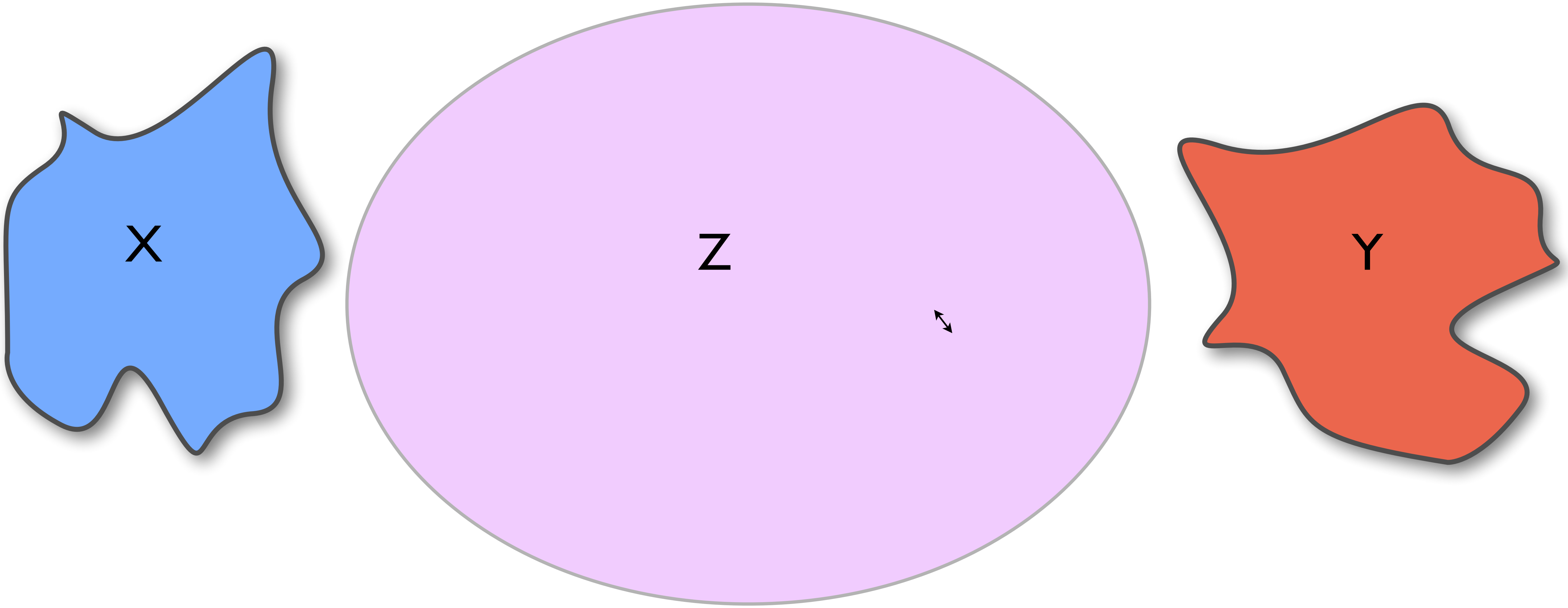

The Gromov–Hausdorff distance between X and Y is the smallest Hausdorff distance between all possible isometric embeddings of X and Y into a *same* metric space Z.

The Brownian tree

↝ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a random compact metric space $\mathcal{T}_{\mathbb{e}}$ such that the convergence

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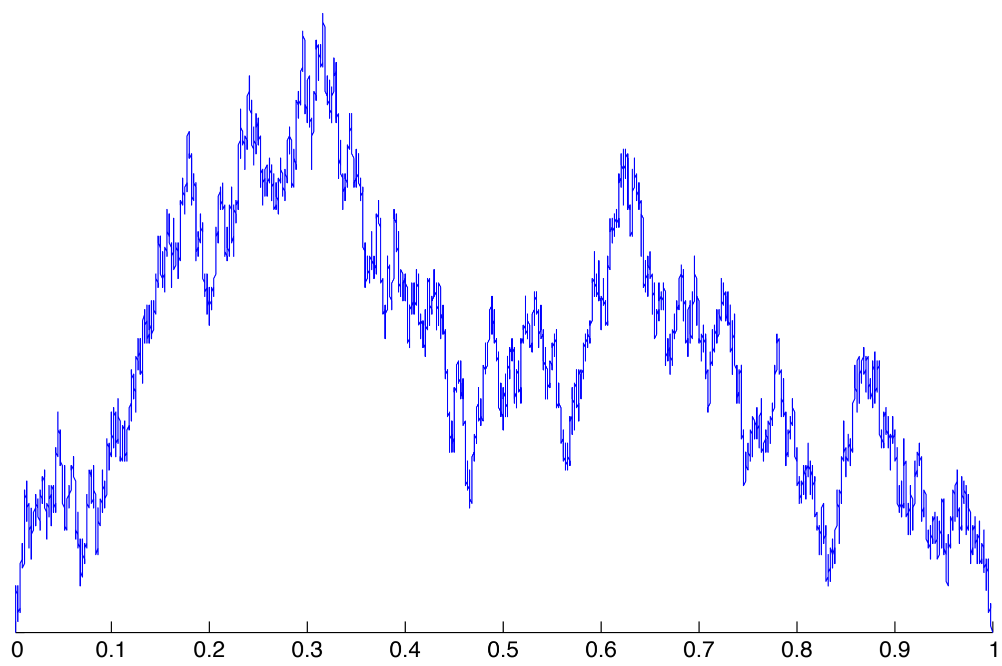

I. Random paths

II. Random trees

III. Random surfaces

construct a random surface as a limit of random discrete surfaces.

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Consider n triangles, and glue them uniformly at random along edges so that one gets a surface homeomorphic to the sphere.

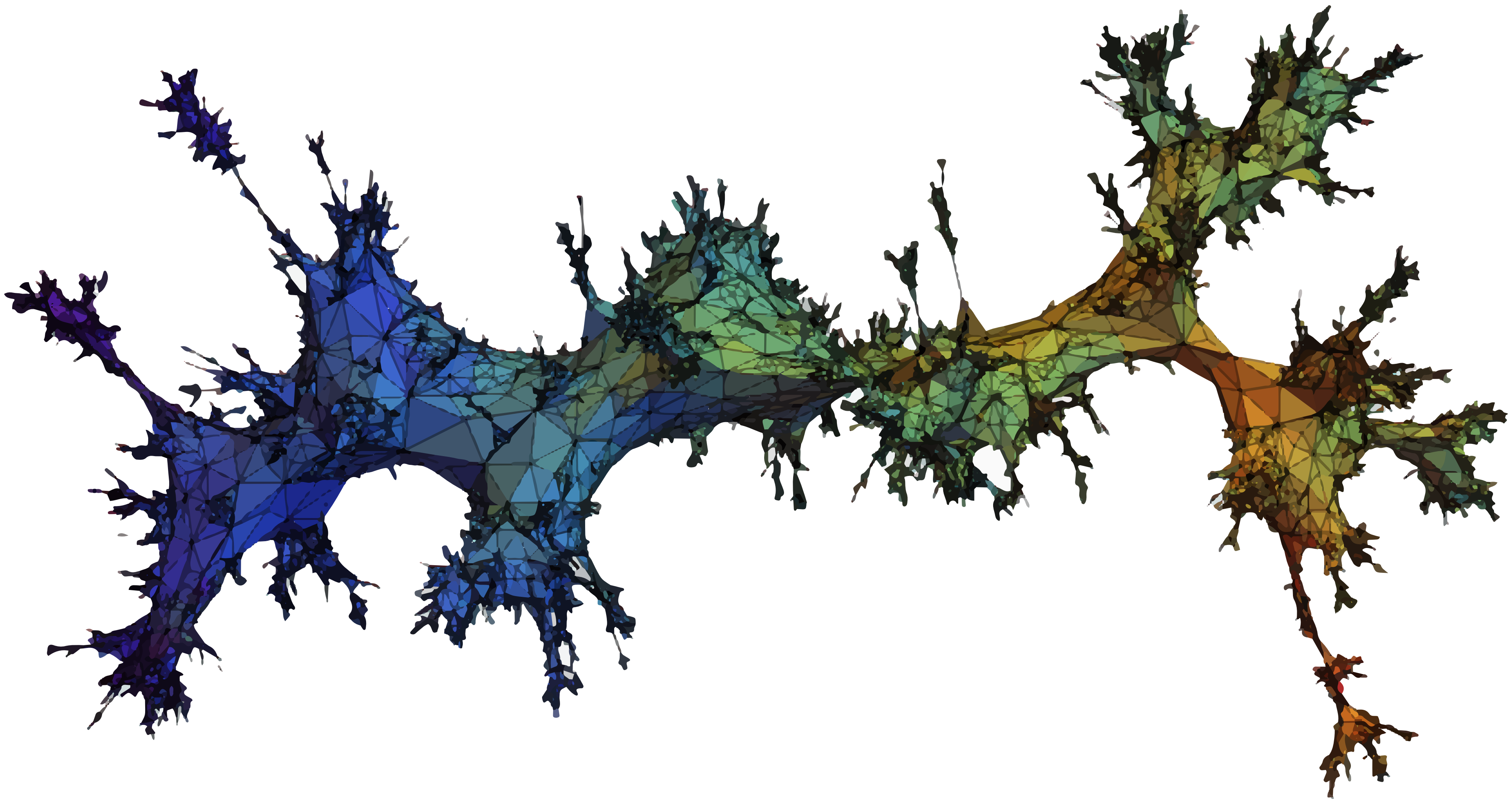
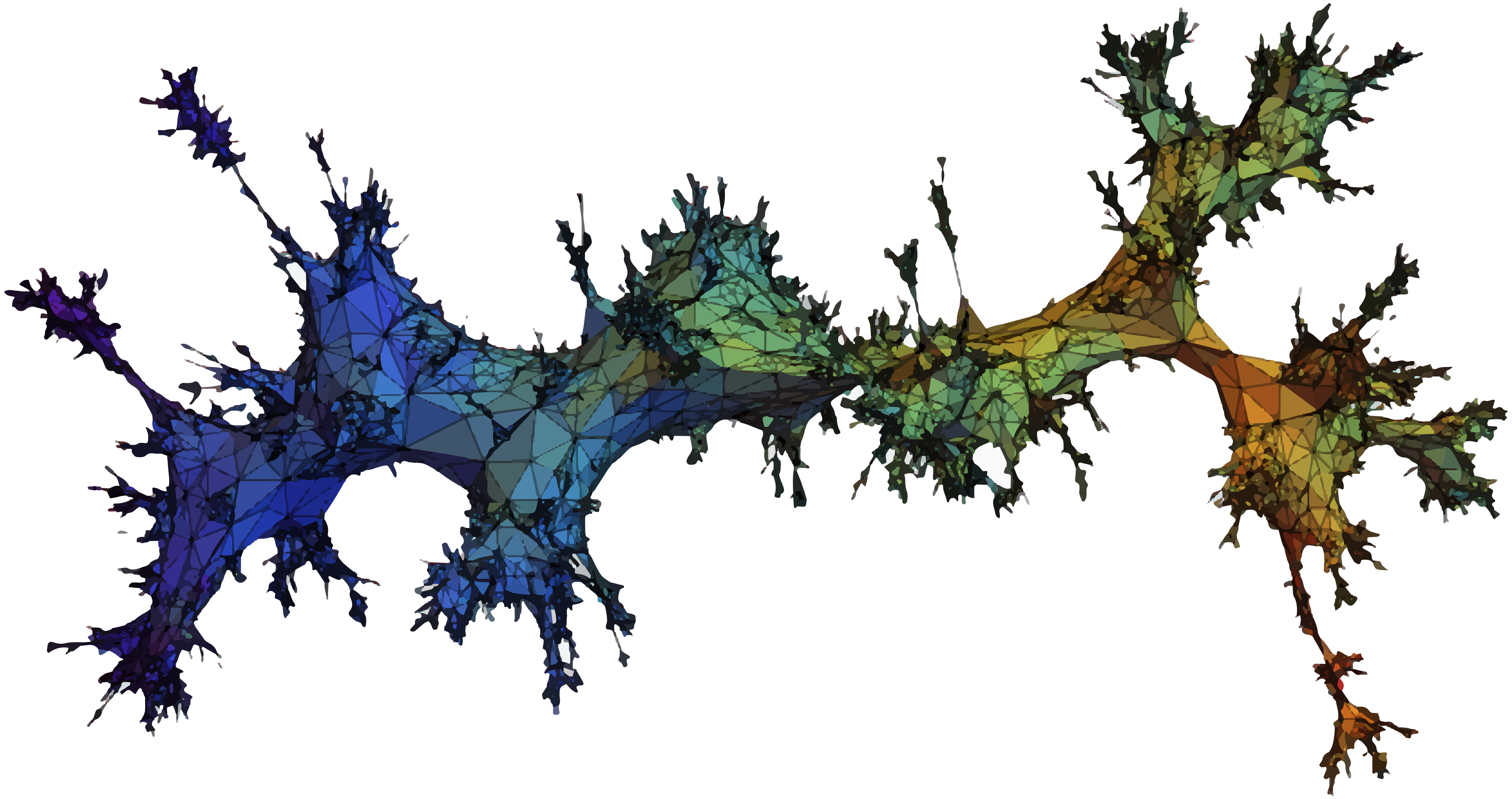


Figure: A large random triangulation of the sphere.



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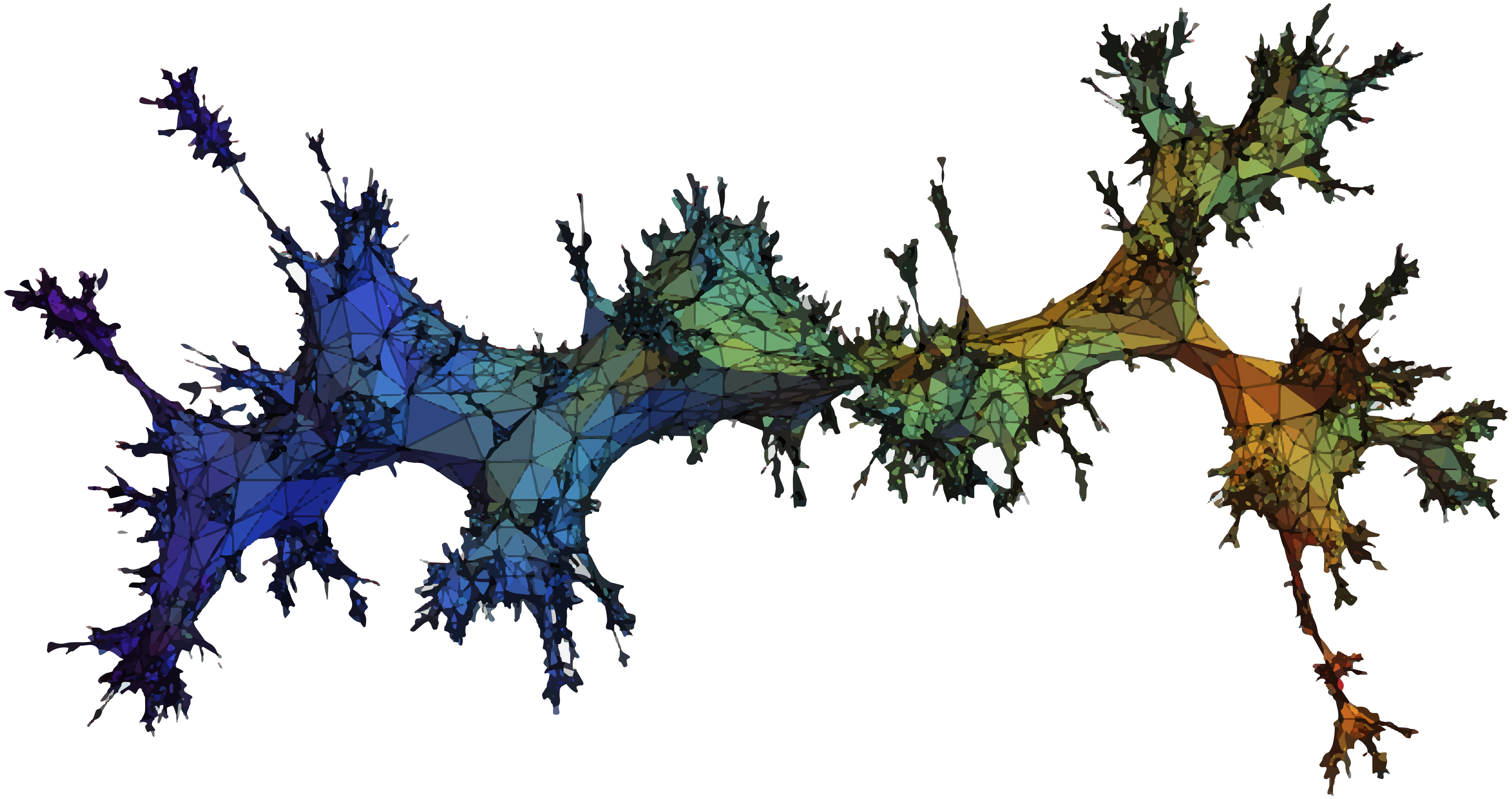

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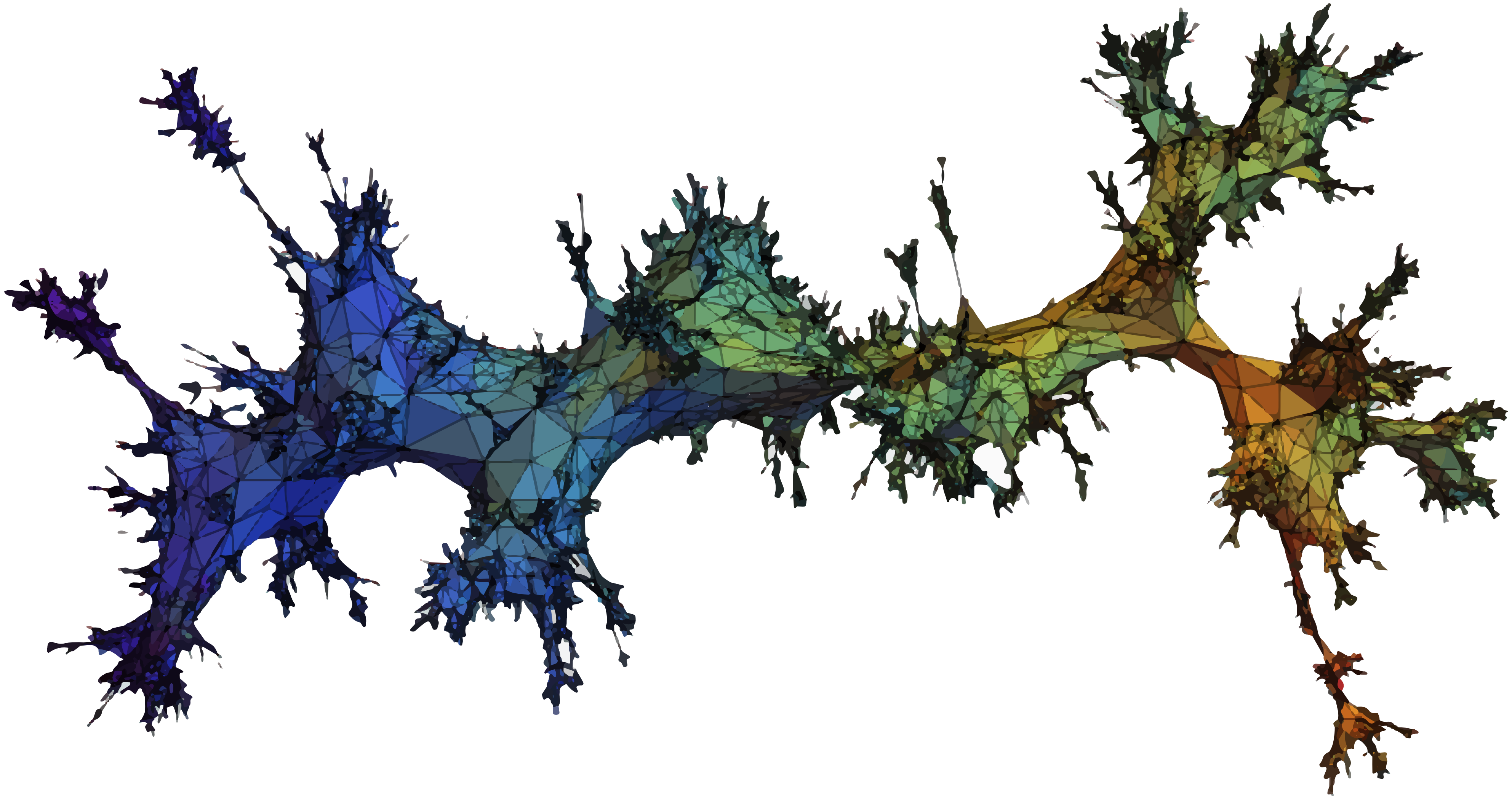

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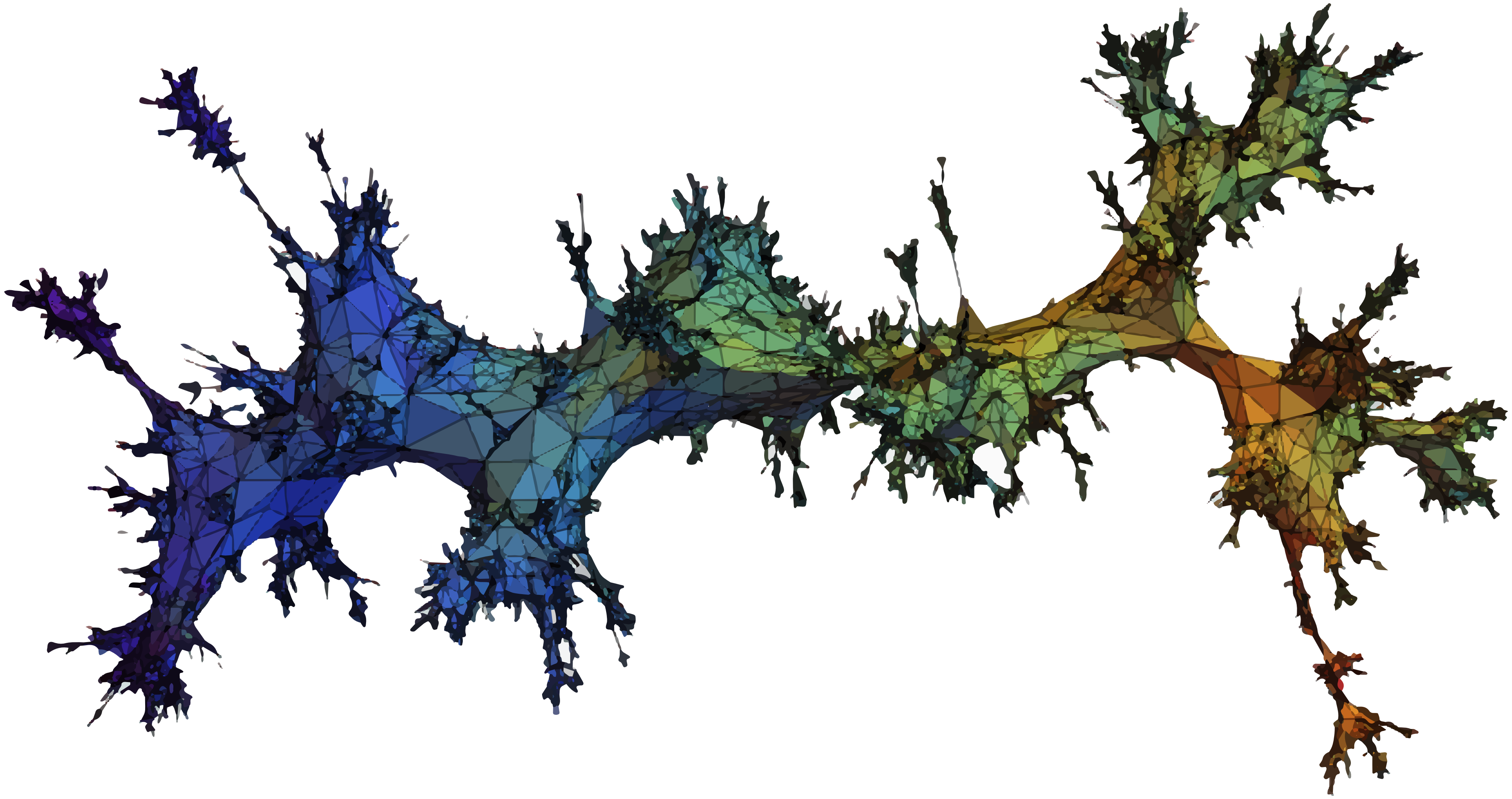

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The Brownian sphere

Problem (Schramm at ICM 2006): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges to a random compact metric space (the Brownian sphere), in distribution for the Gromov–Hausdorff distance.

Solves by Le Gall in 2011.

Since, it has been shown that many other models of discrete random surfaces converge to the Brownian sphere (Miermont, Beltran & Le Gall, Addario-Berry & Albenque, Bettinelli & Jacob & Miermont, Abraham), by using various techniques (in particular bijective codings by labeled trees)

$\leadsto$ Motivations from theoretical physics (Liouville quantum gravity), links with Gaussian multiplicative chaos.

Compact Brownian surfaces II. Orientable surfaces

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Abstract

Fix an arbitrary compact orientable surface with a boundary and consider a uniform bipartite random quadrangulation of this surface with n faces and boundary component lengths of order $\sqrt{n}$ or of lower order. Endow this quadrangulation with the usual graph metric renormalized by $n^{-1/4}$, mark it on each boundary component, and endow it with the counting measure on its vertex set renormalized by n^{-1}, as well as the counting measure on each boundary component renormalized by $n^{-1/2}$. We show that, as $n \to \infty$, this random marked measured metric space converges in distribution for the Gromov–Hausdorff–Prokhorov topology, toward a random limiting marked measured metric space called a *Brownian surface*.

This extends known convergence results of uniform random planar quadrangulations with at most one boundary component toward the *Brownian sphere* and toward the *Brownian disk*, by considering the case of quadrangulations on general compact orientable surfaces. Our approach consists in cutting a Brownian surface into elementary pieces that are naturally related to the Brownian sphere and the Brownian disk and their noncompact analogs, the Brownian plane and the Brownian half-plane, and to prove convergence results for these elementary pieces, which are of independent interest.