Chapter 0: basic discrete probability (warm-up)

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The function P is called a probability measure on Ω.

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called the conditional probability measure given A.

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so A and B are independent.

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One often just writes $P(X = x)$ or $P(\{X = x\})$.

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Two random variables X_1 and X_2 are independent if for every x_1, x_2:

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Similarly, one says that k random variables $X_1, \ldots, X_k$ are independent if for all $x_1, \ldots, x_k$:

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This corresponds to the fact that $X_1, \ldots, X_k$ could be viewed as the outcomes of totally independent experiments.

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The previous events $A = \{(i,j) \in \Omega : i \geqslant 4\}$ and $B = \{(i,j) \in \Omega : j = 6\}$ correspond to the fact that the outcome of the first one is greater or equal to 4 while the event B corresponds to the fact that the outcome of the second one is a 6.

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