Chapter 5: Uniformly integrable martingales

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Outline: 1) Recap on uniform integrability (UI)
2) UI martingales
3) Optional stopping

1) Recap on UI

Recall that a family $(X_i)_{i \in I}$ of real-valued r.v is UI if $\sup_{i \in I} \mathbb{E}[|X_i| \mathbb{1}_{|X_i| \geq K}] \xrightarrow[K \to \infty]{} 0$, and that it is $\Leftrightarrow$ to $\sup_{i \in I} \mathbb{E}[|X_i|] < \infty$ and $\forall \varepsilon > 0, \exists \delta > 0 : \mathbb{P}(A) \leq \delta \Rightarrow \forall i \in I \; \mathbb{E}[|X_i| \mathbb{1}_A] \leq \varepsilon$.

We saw that $X_n \xrightarrow{L^1} X$ iff $X_n \xrightarrow{\mathbb{P}} X$ and (X_n) UI (sometimes called "super dominated convergence")

Theorem (Strong law of large numbers: a.s. & L^1)
Let $(X_n)_{n \geq 1}$ be iid integrable r.v. Then $\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n \to \infty]{} \mathbb{E}[X_1]$ a.s and in L^1

Proof Set $Z_n = \frac{X_1 + \cdots + X_n}{n}$. We already proved that $Z_n \xrightarrow{a.s} \mathbb{E}[X_1]$. Thus $Z_n \xrightarrow{\mathbb{P}} \mathbb{E}[X_1]$. To show L^1 convergence, it thus suffices to check that (Z_n) is UI. We use the ε-δ criterion.

- $\mathbb{E}[|Z_n|] \leq \sum_{k=1}^{n} \frac{\mathbb{E}[|X_k|]}{n} = \mathbb{E}[|X_1|]$, so $(Z_n)_{n \geq 1}$ is bounded in L^1.
- Fix $\varepsilon > 0$. Since $X_1 \in L^1$, the family $(X_k)_{k \geq 1}$ is UI, so $\exists \delta > 0$ s.t $\mathbb{P}(A) \leq \delta \Rightarrow \forall i \geq 1, \mathbb{E}[|X_i| \mathbb{1}_A] \leq \varepsilon$. Then
$\mathbb{E}[|Z_n| \mathbb{1}_A] \leq \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[|X_k| \mathbb{1}_A] \leq \varepsilon$

This completes the proof

Proposition Take $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{A}_i)_{i\in I}$ be a collection of σ-fields $\subset \mathcal{F}$. Then $(\mathbb{E}[X|\mathcal{A}_i])_{i\in I}$ is UI

Proof By writing $X = X^+ - X^-$ with $X^+, X^- \geq 0$, and since the sum of two UI families is UI we may assume that $X \geq 0$. We show that $\sup_{i\in I} \mathbb{E}[\,\mathbb{E}[X|\mathcal{A}_i]\,\mathbb{1}_{\mathbb{E}[X|\mathcal{A}_i]\geq K}] \xrightarrow[K\to\infty]{} 0$

Fix $\varepsilon > 0$. Since $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $\exists \delta > 0$ s.t. $\mathbb{P}(A) \leq \delta \Rightarrow \mathbb{E}[X \mathbb{1}_A] \leq \varepsilon$

Now choose $K > 0$ such that $K \geq \frac{\mathbb{E}[X]}{\delta}$ and write

$$\mathbb{E}[\mathbb{E}[X|\mathcal{A}_i]\,\underbrace{\mathbb{1}_{\mathbb{E}[X|\mathcal{A}_i]\geq K}}_{\mathcal{A}_i \text{ measurable}}] = \mathbb{E}[X\,\mathbb{1}_{\mathbb{E}[X|\mathcal{A}_i]\geq K}] \leq \varepsilon$$

because $\mathbb{P}(\mathbb{E}[X|\mathcal{A}_i] \geq K) \leq \dfrac{\mathbb{E}[\mathbb{E}[X|\mathcal{A}_i]]}{K} = \dfrac{\mathbb{E}[X]}{K} \leq \delta$

This shows the result

2) UI martingales

Here $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}$ is a filtration

Theorem Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$-martingale. The following are $\Leftrightarrow$:

1. $(M_n)_{n\geq 0}$ converges a.s and in L^1 to a r.v denoted by M_∞
2. $\exists X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ s.t $\forall n \geq 0$, $M_n = \mathbb{E}[X|\mathcal{F}_n]$
3. (M_n) is UI

If these conditions hold, one may take $X = M_\infty$ in 2. We say that (M_n) is a closed martingale.

Proof 1 $\Rightarrow$ 2 Fix $n \geq 1$. We know that for $p \geq n$, $\mathbb{E}[M_p|\mathcal{F}_n] = M_n$

Since $|\mathbb{E}[M_\infty|\mathcal{F}_n] - \mathbb{E}[M_p|\mathcal{F}_n]| \leq \mathbb{E}[|M_\infty - M_p|\,|\mathcal{F}_n]$, it follows that

$$\mathbb{E}[\,|\mathbb{E}[M_\infty|\mathcal{F}_n] - M_n|\,] \leq \mathbb{E}[|M_\infty - M_p|] \xrightarrow[p\to\infty]{} 0,$$

so $M_n = \mathbb{E}[M_\infty|\mathcal{F}_n]$

2 $\Rightarrow$ 3 We have already seen that the family $(\mathbb{E}[X|\mathcal{F}_n])_{n\geq 1}$ is UI.

3 $\Rightarrow$ 1 : If (M_n) is UI, then (M_n) is bounded in L^1. So (M_n) convergences a.s, and thus in probability. Since (M_n) is UI, it converges in L^1

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Corollary Let $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. The martingale $M_n = \mathbb{E}[Z | \mathcal{F}_n]$ converges a.s. and in L^1 to $M_\infty = \mathbb{E}[Z|\mathcal{F}_\infty]$ where $\mathcal{F}_\infty = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$

Proof: By the theorem M_n converges a.s and in L^1 to a limit M_∞. We show that $M_\infty = \mathbb{E}[Z|\mathcal{F}_\infty]$.

- M_∞ is $\mathcal{F}_\infty$ measurable since all the M_n are. M_∞ is a L^1 limit, so M_∞ is integrable
- We check that $\mathbb{E}[M_\infty \mathbb{1}_A] = \mathbb{E}[Z \mathbb{1}_A]$ for every $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, which is a generating π-system of $\mathcal{F}_\infty$ containing Ω. This will imply the result. (see Exercise 2 Sheet 8)

Take $A \in \mathcal{F}_n$ and $p \geq n$. Then $\mathbb{E}[Z \mathbb{1}_A] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_p] \mathbb{1}_A]$ because $\mathbb{1}_A$ is $\mathcal{F}_n$ thus $\mathcal{F}_p$ measurable ($\mathcal{F}_n \subset \mathcal{F}_p$)

$$= \mathbb{E}[M_p \mathbb{1}_A] \xrightarrow[p\to\infty]{} \mathbb{E}[M_\infty \mathbb{1}_A].$$

Indeed, $|\mathbb{E}[M_p \mathbb{1}_A] - \mathbb{E}[M_\infty \mathbb{1}_A]| \leq \mathbb{E}[|M_p - M_\infty|] \xrightarrow[p\to\infty]{} 0$.

So $\mathbb{E}[Z\mathbb{1}_A] = \mathbb{E}[M_\infty \mathbb{1}_A]$

3) Optional stopping

The goal is to study whether $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ for (M_n) martingale and T random.

Definition A r.v. $T : (\Omega, \mathcal{F}) \to \mathbb{N} \cup \{+\infty\}$ is called a stopping time (with respect to $(\mathcal{F}_n)$) if for every $n \geq 0$ we have $\{T = n\} \in \mathcal{F}_n$. It is said to be finite if $T < \infty$ a.s.

Remarks
- T is a stopping time iff $\forall n \geq 0$ $\{T \leq n\} \in \mathcal{F}_n$ iff $\forall n \geq 0$ $\{T > n\} \in \mathcal{F}_n$

  (write $\{T \leq n\} = \bigcup_{i=0}^{n} \{T = i\}$, $\{T > n\} = \Omega \setminus \{T \leq n\}$)
- $\{T = +\infty\} = \Omega \setminus \bigcup_{i=0}^{\infty} \{T = i\} \in \mathcal{F}_\infty$ where $\mathcal{F}_\infty = \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$

Interpretation: in the "game" interpretation, stopping times are the random times at which we can decide to stop to play ("without looking at the future")

Examples: ① If $k \geq 0$ the constant time $T = k$ is a stopping time

② If X_n is $\mathcal{F}_n$-measurable and $A \in \mathcal{B}(\mathbb{R})$, then $T_A = \inf\{n \geq 0 : X_n \in A\}$ (with the convention $\inf \emptyset = \infty$) is a stopping time, called the hitting time of A. Indeed,

$$\{T_A = n\} = \{X_0 \notin A, \dots, X_{n-1} \notin A, X_n \in A\} \in \mathcal{F}_n$$

However, in general, for $N>0$, $\sup\{n \le N : Y_n \in A\}$ is not a stopping time

Lemma Let (M_n) be a $(\mathcal{F}_n)$ martingale and T a stopping time.
Then the so-called stopped process $(M_{n\wedge T})_{n\geq 0}$ is a martingale (here $n\wedge T = \min(n,T)$ and $(M_{n\wedge T})(\omega) = M_{n\wedge T(\omega)}(\omega)$)

Proof: • For $n\geq 0$, $M_{n\wedge T} = \sum_{j=0}^{n} \mathbb{1}_{T=j} M_j + \mathbb{1}_{T>n} M_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$

$$\begin{aligned}
\bullet\ \mathbb{E}[M_{(n+1)\wedge T} - M_{n\wedge T} | \mathcal{F}_n] &= \mathbb{E}[\mathbb{1}_{T>n}(M_{n+1}-M_n)|\mathcal{F}_n] \\
&= \mathbb{1}_{T>n}\, \mathbb{E}[M_{n+1}-M_n | \mathcal{F}_n] \\
&= \mathbb{1}_{T>n} (\mathbb{E}[M_{n+1}|\mathcal{F}_n] - M_n) \\
&= 0
\end{aligned}$$

Thus $\mathbb{E}[M_{(n+1)\wedge T}|\mathcal{F}_n] = \mathbb{E}[M_{n\wedge T}|\mathcal{F}_n] = M_{n\wedge T}$.

$\backsim$

Corollary (same assumption) For every $n\geq 0$, $\mathbb{E}[M_{n\wedge T}] = \mathbb{E}[M_0]$

Our goal is to try to get rid of "n" and get $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. This is not true in general:

Let $(X_n)_{n\geq 0}$ be iid random variables with $\mathbb{P}(X_1 = \pm 1) = \frac{1}{2}$, and set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n\geq 1$.
Set $\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(S_0, \dots, S_n)$ for $n\geq 0$.
We have already seen that $(S_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$ martingale
Set $T = \inf\{n\geq 0 : S_n = -1\}$. We will later see that $T<\infty$ a.s., so a.s $S_T = -1$, so that
$\mathbb{E}[S_T] = -1 \neq 0 = \mathbb{E}[S_0]$.

But it is possible to find sufficient conditions to have $\mathbb{E}[M_T] = \mathbb{E}[M_0]$: it is the goal of the optional sampling theorems.

Definition For a stopping time, set
$$\mathcal{F}_T = \{ A \in \mathcal{F} : \forall n\geq 0,\ A \cap \{T=n\} \in \mathcal{F}_n \}$$

It is a simple matter to check that $\mathcal{F}_T$ is a σ-field and $\mathcal{F}_T = \mathcal{F}_n$ when T is constant $= n$

Interpretation: $\mathcal{F}_T$ is the information about what happened until time T

Lemma Assume that M_n is $\mathcal{F}_n$ measurable, T a stopping time

① Assume that $T<\infty$ a.s. Set $M_T = \sum_{n=0}^{\infty} \mathbb{1}_{\{T=n\}} M_n$ (so $M_T = 0$ if $T=\infty$). Then M_T is $\mathcal{F}_T$ measurable

② Assume that $M_n \xrightarrow{a.s.} M_\infty$. Set $M_T = \sum_{n=0}^{\infty} \mathbb{1}_{\{T=n\}} M_n + \mathbb{1}_{\{T=\infty\}} M_\infty$. Then M_T is $\mathcal{F}_T$ measurable

Proof ① We check that $\forall n \geq 0$, $\mathbb{1}_{T=n} M_n$ is $\mathcal{F}_T$ measurable.

To this end, take $B \in \mathcal{B}(\mathbb{R})$ and $n \geq 0$. We show that $\{\mathbb{1}_{T=n} M_n \in B\} \in \mathcal{F}_T$. First, $\{\mathbb{1}_{T=n} M_n \in B\} \in \mathcal{F}$ since $\mathbb{1}_{T=n} M_n$ is $\mathcal{F}_n$-measurable.

Assume $0 \notin B$. Take $p \geq 0$ and write
$$\{\mathbb{1}_{T=n} M_n \in B\} \cap \{T=p\} = \begin{cases} \emptyset \in \mathcal{F}_p & \text{if } p \neq n \\ \{M_n \in B\} \cap \{T=n\} \in \mathcal{F}_n & \text{if } p=n \end{cases}$$

If $0 \in B$, write $\{\mathbb{1}_{T=n} M_n \in B\} = \{\mathbb{1}_{T=n} M_n \in B^c\}^c \in \mathcal{F}_T$.

② By ① it suffices to check that $\mathbb{1}_{T=\infty} M_\infty$ is $\mathcal{F}_T$ measurable. Similarly, for $B \in \mathcal{B}(\mathbb{R})$,

$\{\mathbb{1}_{T=\infty} M_\infty \in B\} \in \mathcal{F}$ because T is $\mathcal{F}$ measurable and M_∞ is $\mathcal{F}$ measurable as an a.s. limit of $\mathcal{F}$ measurable r.v.

For $0 \notin B$ and $p \geq 0$, $\{\mathbb{1}_{T=\infty} M_\infty \in B\} \cap \{T=p\} = \emptyset \in \mathcal{F}_p$

For $0 \in B$ and $p \geq 0$, $\{\mathbb{1}_{T=\infty} M_\infty \in B\} = \{\mathbb{1}_{T=\infty} M_\infty \in B^c\}^c \in \mathcal{F}_T$

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Remark If $T<\infty$ a.s, then $M_{n\wedge T} \xrightarrow{a.s} M_T$. Indeed, when $T<\infty$, for n sufficiently large $M_{n\wedge T} = M_T$, so $M_{n\wedge T} \xrightarrow[n\to\infty]{} M_T$

Theorem (optional stopping)

Let (M_n) be a UI martingale. Denote by M_∞ its a.s and L^1 limit. Let T be a stopping time.

Then $M_T = \mathbb{E}[M_\infty | \mathcal{F}_T]$

In particular $\mathbb{E}[M_T] = \mathbb{E}[M_\infty] = \mathbb{E}[M_n]$ for every $n \geq 0$

Proof. • We have already seen that M_T is $\mathcal{F}_T$ - measurable

• Let's check that M_T is integrable

Recall the notation $\mathcal{F}_\infty = \sigma\left(\bigcup_{n\geq 0} \mathcal{F}_n\right)$ and note that M_∞ is $\mathcal{F}_\infty$ measurable as an a.s. limit of $\mathcal{F}_\infty$ measurable r.v.,

so that $\mathbb{E}[M_\infty | \mathcal{F}_\infty] = M_\infty$.

Write:

$$\mathbb{E}[|M_T|] = \sum_{n\in\mathbb{N}\cup\{+\infty\}} \mathbb{E}[\mathbb{1}_{T=n} |M_n|] = \sum_{n\in\mathbb{N}\cup\{\infty\}} \mathbb{E}[\mathbb{1}_{T=n} | \mathbb{E}[M_\infty|\mathcal{F}_n]|] \leq \sum_{n\in\mathbb{N}\cup\{+\infty\}} \mathbb{E}[\mathbb{1}_{T=n} \mathbb{E}[|M_\infty| | \mathcal{F}_n]]$$

$$= \sum_{n\in\mathbb{N}\cup\{+\infty\}} \mathbb{E}[\mathbb{1}_{T=n} |M_\infty|]$$

$$= \mathbb{E}[|M_\infty|] < \infty.$$

• For $A \in \mathcal{F}_T$, $\mathbb{E}[\mathbb{1}_A M_T] = \sum_{n\in\mathbb{N}\cup\{+\infty\}} \mathbb{E}[\mathbb{1}_{A\cap\{T=n\}} M_T]$

$$= \sum_{n\in\mathbb{N}\cup\{+\infty\}} \mathbb{E}[\mathbb{1}_{A\cap\{T=n\}} M_n]$$

$$= \sum_{n\in\mathbb{N}\cup\{+\infty\}} \mathbb{E}[\mathbb{1}_{A\cap\{T=n\}} \mathbb{E}[M_\infty|\mathcal{F}_n]]$$

$$= \sum_{n\in\mathbb{N}\cup\{+\infty\}} \mathbb{E}[\mathbb{1}_{A\cap\{T=n\}} M_\infty]$$

$$= \mathbb{E}[\mathbb{1}_A M_\infty]$$

This shows that $M_T = \mathbb{E}[M_\infty | \mathcal{F}_T]$

Corollary If (M_n) is a martingale, T a finite stopping time such that $(M_{n\wedge T})_{n\geq 0}$ is UI.

Then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$

(apply the theorem with the stopped martingale $(M_{n\wedge T})_{n\geq 0}$, which converges a.s to M_T)

Example If $(M_{n\wedge T})_{n\geq 0}$ is bounded, then $(M_{n\wedge T})_{n\geq 0}$ is UI.

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⚠ the UI assertion is crucial. Let us give an example of a stopped martingale which is not UI

Let $(X_n)_{n\geq 0}$ be iid random variables with $\mathbb{P}(X_1 = \pm 1) = \frac{1}{2}$, and set $S_0 = 0$, $S_n = X_1 + \dots + X_n$ for $n \geq 1$.

Set $\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(S_0, \dots, S_n)$ for $n \geq 0$.

We have already seen that $(S_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$ martingale

For $x \in \mathbb{Z}$ we set $T_x = \inf\{n \geq 0 : S_n = x\}$ (with the usual convention $\inf \emptyset = \infty$), which is a $(\tilde{\mathcal{F}}_n)$ stopping time. We will see below that $T_x < \infty$ a-s.

Then $S_{n \wedge T_{-1}} \xrightarrow[n\to\infty]{} S_{T_{-1}} = -1$, but $0 = \mathbb{E}[S_{n\wedge T_{-1}}] \not\to 1$, so $(S_{n\wedge T_{-1}})_{n\geq 0}$ is not UI.

(the "spike" comes from the fact that S_n takes very high values when $n < T_{-1}$)

We illustrate the use of the optional stopping theorem by studying $(S_n)_{n\geq 0}$.

For $a < 0 < b$ we set $T_{a,b} = \min(T_a, T_b)$

Proposition Fix $x \in \mathbb{Z}$ and $a < 0 < b$.

① $\mathbb{P}(T_a < T_b) = \frac{b}{b-a}$

② a.s $T_x < \infty$

③ $\mathbb{E}[T_{a,b}] = |ab|$

④ for $u \geq 0$ $\mathbb{E}[e^{-uT_b}] = e^{-\cosh^{-1}(\exp(u))\, b}$ with $\cosh x = \frac{e^x + e^{-x}}{2}$.

Proof: ① $T_{a,b}$ is a stopping time. Let us check that it is finite. Clearly, if (S_n) makes $|a|+b$ "+1" steps in a row, we have $T_{a,b} < \infty$. Thus it is enough to check that a.s (S_n) makes $|a|+b$ "+1" in a row. Set $k = |a| + b$ and let $(A_i)_{i\geq 1}$ be events defined by

$A_1 = \{X_1 = \cdots = X_k = +1\}$, $A_2 = \{X_{k+1} = \cdots = X_{2k} = +1\}, \ldots, A_i = \{X_{ki+1} = \cdots = X_{ki} = +1\}, \ldots$ (block-type argument).

Then $(A_i)_{i\geq 1}$ are $\perp\!\!\!\perp$ (coalition principle) and $\mathbb{P}(A_i) = \frac{1}{2^k}$ for every $i \geq 1$. Thus $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty$ and by Borel-Cantelli 2 a.s A_i happens infinitely often.

This shows that $T_{a,b} < \infty$ a.s.

$(M_{n\wedge T_{a,b}})_{n\geq 0}$ is UI since it is bounded by $|a|+b$. Thus $\mathbb{E}[M_{T_{a,b}}] = \mathbb{E}[M_0] = 0$,

so $0 = a\,\mathbb{P}(T_a < T_b) + b\,\underbrace{\mathbb{P}(T_a > T_b)}_{(1-\mathbb{P}(T_a < T_b))}$, which gives the result.

② Observe that $T_b \geq b$, so $T_b \xrightarrow[b\to\infty]{} +\infty$. Also, (T_b) is $\nearrow$. Thus $\mathbb{P}(T_a < T_b) \xrightarrow[b\to\infty]{} \mathbb{P}(T_a < \infty)$

But $\frac{b}{b-a} \xrightarrow[b\to\infty]{} 1$.

So a.s $T_a < \infty$. By symmetry, $T_{-a} < \infty$ a.s.

(3) The idea is to consider $Q_n = S_n^2 - n$, which is a martingale (called quadratic martingale). Indeed:

- $Q_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$
- $\mathbb{E}[Q_{n+1} | \mathcal{F}_n] = \mathbb{E}[(S_n + X_{n+1})^2 - (n+1) | \mathcal{F}_n]$

$$= \mathbb{E}[S_n^2 + 2 S_n X_{n+1} + X_{n+1}^2 | \mathcal{F}_n] - (n+1)$$

$$= S_n^2 + 2 S_n \mathbb{E}[X_{n+1}] + \mathbb{E}[X_{n+1}^2] - (n+1)$$

$$= S_n^2 - n = Q_n$$

$(Q_{n \wedge T_{a,b}})_{n \geq 0}$ is a martingale. It is not immediate to see that it is UI, so we argue directly to show that $\mathbb{E}[Q_{T_{a,b}}] = 0$.

Indeed, $\mathbb{E}[Q_{n \wedge T_{a,b}}] = 0$, so $\mathbb{E}[S^2_{T_{a,b} \wedge n}] = \mathbb{E}[T_{a,b} \wedge n]$

But $\mathbb{E}[S^2_{T_{a,b} \wedge n}] \xrightarrow[n \to \infty]{} \mathbb{E}[S^2_{T_{a,b}}]$ by dominated convergence ($S^2_{T_{a,b} \wedge n} \leq a^2 + b^2$)

and $\mathbb{E}[T_{a,b} \wedge n] \xrightarrow[n \to \infty]{} \mathbb{E}[T_{a,b}]$ by monotone convergence

We conclude that $\mathbb{E}[T_{a,b}] = \mathbb{E}[S^2_{T_{a,b}}] = a^2 \, \mathbb{P}(T_a < T_b) + b^2 \, \mathbb{P}(T_b < T_a) = |ab|$

(4) The idea is to consider for $\lambda \in \mathbb{R}$ $M_n = \dfrac{e^{\lambda S_n}}{(\cosh \lambda)^n}$, which is a martingale (called exponential martingales). Indeed

- $M_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$
- $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \dfrac{1}{(\cosh \lambda)^{n+1}} \mathbb{E}[e^{\lambda S_n + \lambda X_{n+1}} | \mathcal{F}_n] = \dfrac{1}{(\cosh \lambda)^{n+1}} e^{\lambda S_n} \underbrace{\mathbb{E}[e^{\lambda X_{n+1}}]}_{= \frac{1}{2}(e^{\lambda} + e^{-\lambda})} = M_n$

Take $\lambda \geq 0$.

Then $(M_{n \wedge T_b})_{n \geq 0}$ is a martingale, which is UI since bounded: $|M_{n \wedge T_b}| \leq e^{\lambda b}$

Thus $1 = \mathbb{E}[M_0] = \mathbb{E}[M_{T_b}] = \mathbb{E}\left[\dfrac{e^{\lambda b}}{(\cosh \lambda)^{T_b}}\right]$, so $\mathbb{E}[(\cosh \lambda)^{-T_b}] = e^{-\lambda b}$,

and the desired result follows by taking u such that $\cosh \lambda = e^{u}$.