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ETH

Chapter 6: L^p martingales $(p>1)$

Outline: 1) Doob maximal inequalities

2) Martingales bounded in L^p $(p>1)$

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_n)_{n\geq 0}$ is a filtration with $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$

1) Doob maximal inequalities

Theorem (Doob maximal inequalities)

① Let (M_n) be a submartingale. Then for every $a>0$ and $n \geq 0$:

$$a\, \mathbb{P}\left(\max_{0\leq k\leq n} M_k \geq a\right) \leq \mathbb{E}\left[M_n \mathbb{1}_{\{\max_{0\leq k\leq n} M_k \geq a\}}\right] \overset{(*)}{\leq} \mathbb{E}[M_n^+] \quad \text{with } M_n^+ = \max(M_n, 0)$$

② Let (M_n) be a martingale and set $M_n^* = \max_{0\leq k\leq n} |M_k|$. Then for every $a>0$ and $n\geq 0$

$$a\, \mathbb{P}(M_n^* \geq a) \leq \mathbb{E}\left[|M_n| \mathbb{1}_{M_n^* \geq a}\right] \leq \mathbb{E}[|M_n|]$$

Remarks

- ② follows immediately from ①: if (M_n) is a martingale, then $(|M_n|)$ is a submartingale
- The inequality $(*)$ just comes from $M_n \mathbb{1}_{\{\max_{0\leq k\leq n} M_k \geq a\}} \leq M_n^+$.
- in ① Markov's inequality gives $a\, \mathbb{P}(\max_{0\leq k\leq n} M_k \geq a) \leq a\, \mathbb{P}(\max_{0\leq k\leq n} M_k^+ \geq a) \leq \mathbb{E}[\max_{0\leq k\leq n} M_k^+]$.

  Since $M_n^+ \leq \max_{0\leq k\leq n} M_k^+$, the theorem gives a better bound.

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Proof of ① The idea is to introduce the stopping time $T = \inf\{n\geq 0 : M_n \geq a\}$ with the usual convention $\inf \emptyset = \infty$.

We have $T\leq n$ iff $\max_{0\leq k\leq n} M_k \geq a$, so $a\, \mathbb{P}\left(\max_{0\leq k\leq n} M_k \geq a\right) = \sum_{k=0}^{n} a\, \mathbb{P}(T=k) = \sum_{k=0}^{n} \mathbb{E}[a \mathbb{1}_{T=k}]$

But $T=k$ implies $M_k \geq a$, so $a\, \mathbb{P}(\max_{0\leq k\leq n} M_k \geq a) \leq \sum_{k=0}^{n} \mathbb{E}[M_k \mathbb{1}_{T=k}]$. But (M_n) is a submartingale, so $\mathbb{E}[M_n | \mathcal{F}_k] \geq M_k$

Thus $\mathbb{E}[M_k \mathbb{1}_{T=k}] \leq \mathbb{E}[\mathbb{E}[M_n|\mathcal{F}_k] \mathbb{1}_{T=k}] =$

$= \mathbb{E}[M_n \mathbb{1}_{T=k}]$ because $\mathbb{1}_{T=k}$ is $\mathcal{F}_k$ measurable (T is a stopping time)

We conclude that $a\,\mathbb{P}(\max_{0\le k\le n} M_k \ge a) \le \sum_{k=0}^{n} \mathbb{E}[M_n \mathbb{1}_{T=k}] = \mathbb{E}[M_n \mathbb{1}_{T\le n}] = \mathbb{E}[M_n \mathbb{1}_{\max_{0\le k\le n} M_k \ge a}]$, which completes the proof.

Theorem (Doob L^p inequalities) Fix $p>1$.

① Let $(M_n)_{n\ge0}$ be a non-negative submartingale. Then for every $n\ge0$ $\mathbb{E}[(\max_{0\le k\le n} M_k)^p] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[(M_n)^p]$

② Let $(M_n)_{n\ge0}$ be a martingale. Then, with $M_n^* = \max_{0\le k\le n} |M_k|$, $\mathbb{E}[(M_n^*)^p] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_n|^p]$

Remarks • As before, ② follows from ①: if $(M_n)_{n\ge0}$ is a martingale, $(|M_n|)$ is a ≥0 submartingale

• $\left(\frac{p}{p-1}\right)^p \xrightarrow[p\to1]{} \infty$, which explains why there is no L^1 Doob inequality

The proof of ① uses two ingredients

Lemma (Hölder's inequality) Let $q>1$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Let X,Y be real-valued r.v with $X\in L^p$ and $Y\in L^q$.
Then $\mathbb{E}[|XY|] \le \mathbb{E}[|X|^p]^{1/p} \cdot \mathbb{E}[|Y|^q]^{1/q}$ (often written as $\|XY\|_1 \le \|X\|_p \cdot \|Y\|_q$)

Proof First note that for $a,b\ge0$ $ab \overset{(*)}{\le} \frac{a^p}{p} + \frac{b^q}{q}$. Indeed, if $a=0$ or $b=0$ it is clearly true. If $a,b>0$, using concavity of $\ln$ write $\ln\left(\frac{1}{p}\cdot a^p + \frac{1}{q} b^q\right) \ge \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) = \ln a + \ln b$, which gives $(*)$

Then, we may assume that $\mathbb{E}[|X|^p]>0$ and $\mathbb{E}[|Y|^q]>0$, otherwise $X=0$ a.s or $Y=0$ a.s and the result is true.

By dividing X by $\mathbb{E}[|X|^p]^{1/p}$ and Y by $\mathbb{E}[|Y|^q]^{1/q}$, we may assume $\mathbb{E}[|X|^p]=1$ and $\mathbb{E}[|Y|^q]=1$

Then write $|XY| \le \frac{|X|^p}{p} + \frac{|Y|^q}{q}$, which gives $\mathbb{E}[|XY|] \le \frac{1}{p} + \frac{1}{q} = 1$.

Lemma (moment-tail)

Let $X\ge0$ be a r.v. Then for every $p>0$ $\mathbb{E}[X^p] = p\int_0^\infty x^{p-1}\,\mathbb{P}(X\ge x)\,dx$

Proof Using Fubini-Tonelli, write $p\int_0^\infty x^{p-1}\mathbb{P}(U\ge x)\,dx = p\int_0^\infty x^{p-1}\mathbb{E}[\mathbb{1}_{U\ge x}]\,dx = \mathbb{E}\left[\int_0^U p x^{p-1}\,dx\right] = \mathbb{E}[U^p]$

Proof of Theorem ① If $\mathbb{E}[M_n^p]=\infty$, there is nothing to prove, so we assume $\mathbb{E}[M_n^p]<\infty$.

Step 1: $M_k \in L^p$. Indeed, for $0\le k\le n$, by Jensen's inequality $\mathbb{E}[(M_k)^p] \le \mathbb{E}[(\mathbb{E}[M_n|\mathcal{F}_k])^p] \le \mathbb{E}[\mathbb{E}[M_n^p|\mathcal{F}_k]] = \mathbb{E}[M_n^p]<\infty$, Step 2 $M_n^* = \max_{0\le k\le n} M_k \in L^p$ (recall $M_k \geq 0$). Indeed, $\mathbb{E}[(M_n^*)^p] \leq \mathbb{E}[\sum_{k=0}^{n} M_k^p] \leq (n+1)\,\mathbb{E}[M_n^p] < \infty$

Step 3. write $\mathbb{E}[(M_n^*)^p] = p\int_0^\infty a^{p-2}\cdot a\,\mathbb{P}(M_n^* \geq a)\,da \leq p\int_0^\infty a^{p-2}\,\mathbb{E}[M_n \mathbb{1}_{\{M_n^* \geq a\}}]\,da$

$$= \mathbb{E}\Big[M_n \int_0^{M_n^*} p\,a^{p-2}\,da\Big] = \frac{p}{p-1}\mathbb{E}[M_n (M_n^*)^{p-1}]$$

Take $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ $\left(q=\frac{p}{p-1}\right)$. By Hölder's inequality we get

$$\frac{p}{p-1}\,\mathbb{E}[M_n (M_n^*)^{p-1}] \leq \mathbb{E}[M_n^p]^{1/p}\,\mathbb{E}[(M_n^*)^{(p-1)q}]^{1/q}$$

We conclude that $\mathbb{E}[(M_n^*)^p] \leq \frac{p}{p-1}\mathbb{E}[M_n^p]^{\frac{1}{p}}\,\mathbb{E}[(M_n^*)^p]^{1-\frac{1}{p}}$, which gives the desired result.

2) Martingales bounded in L^p $(p>1)$

We start with recalling that $\mathbb{E}[|X|] \leq \mathbb{E}[|X|^p]^{1/p}$ for $p>1$. So (X_n) bounded in $L^p \Rightarrow (X_n)$ bounded in L^1 and $X_n \xrightarrow{L^p} X$ implies $X_n \xrightarrow{L^1} X$

Theorem Let (M_n) be a martingale. Let $p>1$. Assume that $\sup_{n\geq 0}\mathbb{E}[|M_n|^p] < \infty$. Then:

① M_n converges a.s and in L^p to a r.v. M_∞.

② Setting $M_\infty^* = \sup_{n\geq 0}|M_n|$, $\mathbb{E}[(M_\infty^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_\infty|^p] = \left(\frac{p}{p-1}\right)^p \sup_{n\geq 0}\mathbb{E}[|M_n|^p]$

Proof: ① Step 1: (M_n) is bounded in L^p, thus in L^1, so it converges a.s. to a r.v M_∞

by Doob's L^p inequality, $\mathbb{E}[(M_n^*)^p] \leq \left(\frac{p}{p-1}\right)\mathbb{E}[|M_n|^p] \leq \left(\frac{p}{p-1}\right)\sup_{n\geq 1}\mathbb{E}[|M_n|^p] < \infty$

Step 2: $M_\infty \in L^p$. Indeed, $M_n^* \nearrow M_\infty^*$, so by monotone convergence we get $\mathbb{E}[(M_\infty^*)^p] \underset{(*)}{\leq} \left(\frac{p}{p-1}\right)\sup_{n\geq 1}\mathbb{E}[|M_n|^p] < \infty$,

and in particular $M_\infty^* \in L^p$. Since $|M_n| \leq M_\infty^*$, we get $|M_\infty| \leq M_\infty^*$, so $M_\infty \in L^p$

Step 3 $M_n \xrightarrow{L^p} M_\infty$. Indeed, $|M_n - M_\infty|^p \xrightarrow{a.s.} 0$ and $|M_n - M_\infty|^p \leq (2\max(|M_n|,|M_\infty|))^p \leq 2^p(|M_n|^p + |M_\infty|^p) \leq 2^{p+1}(M_\infty^*)^p \in L^1$,

$\mathbb{E}[|M_n - M_\infty|^p] \xrightarrow[n\to\infty]{} 0.$

② By $(*)$ we just have to check that $\mathbb{E}[|M_\infty|^p] = \sup_{n\geq 0}\mathbb{E}[|M_n|^p]$

Since $(|M_n|^p)_{n\geq 0}$ is a submartingale, $(\mathbb{E}[|M_n|^p])_{n\geq 1}$ is $\uparrow$, so

$$\mathbb{E}[|M_\infty|^p] = \lim_{n\to\infty} \mathbb{E}[|M_n|^p] \quad \text{by } L^p \text{ convergence}$$

$$= \sup_{n\geq 0} \mathbb{E}[|M_n|^p] \quad \text{because } (\mathbb{E}[|M_n|^p])_{n\geq 1} \text{ is } \uparrow$$