Mathematical Foundations for Finance

Exercise sheet 1

Exercise 1.1 Let Ω be a non-empty set.

(a) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Prove that $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a σ-algebra on Ω.

(b) Let $\mathcal{A}$ be a family of subsets of Ω. Show that there is a (clearly unique) minimal σ-algebra $\sigma(\mathcal{A})$ containing $\mathcal{A}$. Here minimality is with respect to inclusion: if $\mathcal{F}$ is a σ-algebra with $\mathcal{A} \subseteq \mathcal{F}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.

(c) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Show by example that $\mathcal{F}_1 \cup \mathcal{F}_2$ may fail to be a σ-algebra.
Hint: You can consider two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega := \{1, 2, 3\}$.

Solution 1.1

(a) We check the requirements for a σ-algebra:

- $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$ because $\Omega \in \mathcal{F}_i$ for all $i \in \{1, 2\}$;
- if $A \in \mathcal{F}_1 \cap \mathcal{F}_2$, then $A \in \mathcal{F}_i$, and hence $A^c \in \mathcal{F}_i$ for all $i \in \{1, 2\}$. It follows that $A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$;
- if $A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$, $n \in \mathbb{N}$, then $A_n \in \mathcal{F}_i$ for all $i \in \{1, 2\}$. Hence, $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_i$ for all $i \in \{1, 2\}$, and thus $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$.

(b) Define

$$\sigma(\mathcal{A}) := \bigcap_{\substack{\mathcal{F}\ \sigma\text{-algebra} \\ \mathcal{F} \supseteq \mathcal{A}}} \mathcal{F}.$$

Notice that the above intersection is over a non-empty family of σ-algebras, since the power set on Ω is a σ-algebra that contains $\mathcal{A}$. Then, the argument in part (a) can be extended to the intersection of an arbitrary family of σ-algebras, proving that $\sigma(\mathcal{A})$ is a σ-algebra, and of course $\sigma(\mathcal{A}) \supseteq \mathcal{A}$. The uniqueness of such a σ-algebra follows immediately from its construction.

(c) Let $\Omega := \{1, 2, 3\}$, and consider the σ-algebras

$$\mathcal{F}_1 := \sigma(\{\{1\}\}) = \{\emptyset, \{1, 2, 3\}, \{1\}, \{2, 3\}\} \text{ and } \mathcal{F}_2 := \sigma(\{\{2\}\}) = \{\emptyset, \{1, 2, 3\}, \{2\}, \{1, 3\}\}.$$

It is straightforward to verify that the union $\mathcal{F}_1 \cup \mathcal{F}_2$ contains both $\{1\}$ and $\{2\}$, yet does not contain $\{1\} \cup \{2\} = \{1, 2\}$. Hence, $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-algebra.

Exercise 1.2 Consider a probability space $(\Omega, \mathcal{F}, P)$. A σ-algebra $\mathcal{F}_0 \subseteq \mathcal{F}$ is said to be *P-trivial* if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{F}_0$. Prove that $\mathcal{F}_0$ is P-trivial if and only if every $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$ is P-a.s. constant.

Solution 1.2 Suppose that $\mathcal{F}_0$ is P-trivial, and consider an $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$. By definition we have that $\{X \leqslant a\} \in \mathcal{F}_0$ for all $a \in \mathbb{R}$, and thus $P[X \leqslant a] \in \{0, 1\}$. Define

$$c := \inf\{a \in \mathbb{R} : P[X \leqslant a] = 1\}.$$

We first prove that $c \in \mathbb{R}$. Since $\{X \leqslant n\} \uparrow \{X \in \mathbb{R}\}$, then $P[X \leqslant n] \uparrow P[X \in \mathbb{R}] = 1$, and so the above infimum is over a nonempty set (i.e. $c \neq \infty$). Then, if $c = -\infty$, we have that $P[X \leqslant -n] = 1$ for all $n \in \mathbb{N}$, and from the fact that $\{X \leqslant -n\} \downarrow \varnothing$, it follows that $1 = \lim_{n\to\infty} P[X \leqslant -n] = P[\varnothing] = 0$. We get the desired contradiction.
By the definition of the infimum, we have that $P[X \leqslant c + \frac{1}{n}] = 1$ and $P[X \leqslant c - \frac{1}{n}] = 0$ for all $n \in \mathbb{N}$. Since $\{X \leqslant c + \frac{1}{n}\} \downarrow \{X \leqslant c\}$ and $\{X \leqslant c - \frac{1}{n}\} \uparrow \{X < c\}$, we get that

$$P[X \leqslant c] = \lim_{n\to\infty} P\left[X \leqslant c + \frac{1}{n}\right] = 1, \text{ and } P[X < c] = \lim_{n\to\infty} P\left[X \leqslant c - \frac{1}{n}\right] = 0.$$

Hence, we conclude that $X = c$ P-a.s. because

$$P[X = c] = P[X \leqslant c] - P[X < c] = 1.$$

Conversely, suppose that every $\mathcal{F}_0$-measurable random variable is P-a.s. constant, and take $A \in \mathcal{F}_0$. Then,

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c \end{cases}$$

is an $\mathcal{F}_0$-measurable random variable, and hence must be P-a.s. constant. It follows immediately that either $P[\mathbb{1}_A = 1] = P[A] = 1$ or $P[\mathbb{1}_A = 0] = P[A^c] = 1$, so that $P[A] \in \{0, 1\}$. This completes the proof.

Exercise 1.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-algebra. Then, the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E[X\mathbb{1}_A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called the *conditional expectation of X given $\mathcal{G}$* and is denoted by $E[X \,|\, \mathcal{G}]$.
[*This is the formal definition of the conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.*]

(a) Show that if X is $\mathcal{G}$-measurable, then $E[X \,|\, \mathcal{G}] = X$ P-a.s.

(b) Show that $E[E[X \,|\, \mathcal{G}]] = E[X]$.

(c) Show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$ (that is, if $\mathcal{G}$ is P-trivial), then $E[X \,|\, \mathcal{G}] = E[X]$ P-a.s.

(d) Consider an integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and some constants $a, b \in \mathbb{R}$. Show that $E[aX + bY \,|\, \mathcal{G}] = aE[X \,|\, \mathcal{G}] + bE[Y \,|\, \mathcal{G}]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e., there exists a collection $(A_i)_{i=1,\dots,n}$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \dots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \dots, n$. Show that

$$E[X \,|\, \mathcal{G}] = \sum_{i=1}^n E[X \,|\, A_i] \, \mathbb{1}_{A_i} \; P\text{-a.s.}$$

This says that the conditional expectation of a random variable given a finitely generated σ-algebra is a *piecewise constant* function with the constants given by the elementary conditional expectations given the sets of the generating partition.

[*This is a very useful property when one conditions on a finitely generated σ-algebra, as for instance in the multinomial model.*]

Hint 1: Recall that $E[X \,|\, A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$ and try to write X as a sum of random variables each of which only takes non-zero values on a single A_i.
Hint 2: Check that any set $A \in \mathcal{G}$ has the form $\cup_{j\in J} A_j$ for some $J \subseteq \{1, \dots, n\}$.

Solution 1.3

(a) X is $\mathcal{G}$-measurable and integrable by assumption, so the first requirement in the definition of conditional expectation is satisfied for $Z = X$. Moreover, we clearly have that $E[X\mathbb{1}_A] = E[X\mathbb{1}_A]$ for all $A \in \mathcal{G}$, hence $E[X\,|\,\mathcal{G}] = X$ P-a.s.

(b) In the definition of the conditional expectation, set $A = \Omega$. Then, we obtain that $E[E[X\,|\,\mathcal{G}]] = E[E[X\,|\,\mathcal{G}]\,\mathbb{1}_\Omega] = E[X\mathbb{1}_\Omega] = E[X]$.

(c) Since $|E[X]| \le E[|X|]$ by Jensen's inequality and $E[|X|] < \infty$ by the assumption that X is integrable, we have that $E[X]$ is integrable as well. $E[X]$ is also trivially $\mathcal{G}$-measurable since it is a constant random variable. Moreover, in this setting, $A \in \mathcal{G}$ only if $P[A] = 0$ or $P[A] = 1$. Noting that

$$\begin{aligned} E[X\mathbb{1}_A] &= 0 = E[E[X]\,\mathbb{1}_A], && \forall A \in \mathcal{G} \text{ such that } P[A] = 0, \\ E[X\mathbb{1}_A] &= E[X] = E[E[X]\,\mathbb{1}_A], && \forall A \in \mathcal{G} \text{ such that } P[A] = 1, \end{aligned}$$

we obtain $E[X\,|\,\mathcal{G}] = E[X]$ P-a.s.

(d) By the definition of the conditional expectation, we have that $E[X\,|\,\mathcal{G}]$ and $E[Y\,|\,\mathcal{G}]$ are $\mathcal{G}$-measurable and integrable; hence, the same holds for $aE[X\,|\,\mathcal{G}] + bE[Y\,|\,\mathcal{G}]$. Choosing some $A \in \mathcal{G}$, we can compute that

$$\begin{aligned} E\big[(aE[X\,|\,\mathcal{G}] + bE[Y\,|\,\mathcal{G}])\mathbb{1}_A\big] &= aE\big[E[X\,|\,\mathcal{G}]\,\mathbb{1}_A\big] + bE\big[E[Y\,|\,\mathcal{G}]\,\mathbb{1}_A\big] \\ &= aE[X\mathbb{1}_A] + bE[Y\mathbb{1}_A] = E[(aX + bY)\mathbb{1}_A], \end{aligned}$$

where the first equality uses the linearity of the (classical) expectation and the second uses the definition of $E[X\,|\,\mathcal{G}]$ and $E[Y\,|\,\mathcal{G}]$. By the arbitrariness of $A \in \mathcal{G}$, we can conclude that $E[aX + bY\,|\,\mathcal{G}] = aE[X\,|\,\mathcal{G}] + bE[Y\,|\,\mathcal{G}]$ P-a.s.

(e) First recall that $E[X\,|\,A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$. Using that

$$X = X\mathbb{1}_\Omega = X\mathbb{1}_{\cup_{i=1}^n A_i} = X\sum_{i=1}^n \mathbb{1}_{A_i} = \sum_{i=1}^n X\mathbb{1}_{A_i},$$

where the third equality holds because A_i are pairwise disjoint, we get by part (d) that

$$E[X\,|\,\mathcal{G}] = \sum_{i=1}^n E[X\mathbb{1}_{A_i}\,|\,\mathcal{G}] \;\; P\text{-a.s.},$$

and hence we only have to show that $E[X\mathbb{1}_{A_i}\,|\,\mathcal{G}] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ P-a.s. for each $i \in \{1, \dots, n\}$. Since $A_i \in \mathcal{G}$ and $E[X\,|\,A_i] = E[X\mathbb{1}_{A_i}]/P[A_i] \in \mathbb{R}$, we already know that $E[X\,|\,A_i]\,\mathbb{1}_{A_i}$ is $\mathcal{G}$-measurable and integrable. One can verify that the family of sets $A = \bigcup_{j\in J} A_j$ for $J \in 2^{\{1,\dots,n\}}$ (the power set of $\{1, \dots, n\}$) forms a σ-field. Let us denote this σ-field by $\tilde{\mathcal{G}}$. Since we clearly have $A_i \in \tilde{\mathcal{G}}$ for all $i \in \{1, \dots, n\}$, we get that $\tilde{\mathcal{G}} \supseteq \mathcal{G}$, which for any $A \in \mathcal{G}$ implies that $A = \bigcup_{j\in J} A_j$ for some $J \subseteq \{1, \dots, n\}$. For any such $A \in \mathcal{G}$, we have that

$$\mathbb{1}_{A_i}\mathbb{1}_A = \begin{cases} \mathbb{1}_{A_i} & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

Hence, we can then compute

$$E\left[\left(\frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}\right)\mathbb{1}_A\right] = \begin{cases} E[X\mathbb{1}_{A_i}]\,\frac{P[A_i]}{P[A_i]} = E[X\mathbb{1}_{A_i}] & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

On the other hand, we have that

$$E\left[X\mathbb{1}_{A_i}\mathbb{1}_A\right] = \begin{cases} E\left[X\mathbb{1}_{A_i}\right] & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

This shows that $E\left[X\mathbb{1}_{A_i}\,|\,\mathcal{G}\right] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ P-a.s. and concludes the proof.

Exercise 1.4 Let $(\Omega, \mathcal{F}, P)$ be the probability space with $\Omega = \{UU, UD, DD, DU\}$, $\mathcal{F} = 2^\Omega$, and P defined by $P[\omega] = 1/4$ for all $\omega \in \Omega$ (so P is the *uniform* probability measure on Ω). Consider the random variables $Y_1, Y_2 \colon \Omega \to \mathbb{R}$ that are given by $Y_1(UU) = Y_1(UD) = 2$, $Y_1(DD) = Y_1(DU) = 1/2$, $Y_2(UU) = Y_2(DU) = 2$, and $Y_2(DD) = Y_2(UD) = 1/2$. Define the process $X = (X_k)_{k=0,1,2}$ by $X_0 = 8$, and $X_k = X_0 \prod_{i=1}^k Y_i$ for $k = 1, 2$.

(a) Show that, for any continuous function $h : \mathbb{R} \to \mathbb{R}$, the composition $h(X_1)$ is $\sigma(X_1)$-measurable.

(b) Draw a tree to illustrate the possible evolutions of the process X from time 0 to time 2, and label the corresponding transition probabilities and probabilities.

(c) Write down the σ-algebras (i.e. give all their sets) defined by $\mathcal{F}_k = \sigma(X_i : 0 \leqslant i \leqslant k)$ and $\mathcal{G}_k = \sigma(X_k)$ for $k = 0, 1, 2$.

(d) Consider the collections of σ-algebras $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ and $\mathbb{G} = (\mathcal{G}_k)_{k=0,1,2}$. Do these form filtrations on $(\Omega, \mathcal{F})$? Why or why not?

(e) If they are indeed filtrations, is X adapted to $\mathbb{F}$ or $\mathbb{G}$?

(f) Give financial interpretations of X, $\mathbb{F}$ and $\mathbb{G}$.

Solution 1.4

(a) It is sufficient to show that $\{h(X_1) \le c\} \in \sigma(X_1)$ for all $c \in \mathbb{R}$. We write

$$\{h(X_1) \le c\} = (h(X_1))^{-1}(-\infty, c] = X_1^{-1}(h^{-1}(-\infty, c]).$$

Since $(-\infty, c]$ is closed and h is continuous, then $h^{-1}(-\infty, c]$ is also closed, and hence $h^{-1}(-\infty, c] \in \mathcal{B}(\mathbb{R})$. Then, X_1 is clearly measurable with respect to $\sigma(X_1)$, and thus we have that $X_1^{-1}(h^{-1}(-\infty, c]) \in \sigma(X_1)$, completing the proof.

(b) The tree with the transition probabilities labelled and the one with the probabilities labelled are drawn below.

(c) Since $X_0 = 8$ is a constant, then $\mathcal{F}_0 = \mathcal{G}_0 = \sigma(X_0) = \{\varnothing, \Omega\}$. We thus also have $\mathcal{F}_1 = \sigma(X_0, X_1) = \sigma(X_1) = \mathcal{G}_1$. Moreover, because X_1 is either 4 or 16, and $X_1^{-1}(4) = \{DD, DU\}$ and $X_1^{-1}(16) = \{UU, UD\}$, we have

$$\sigma(X_1) = \{\varnothing, \Omega, \{DD, DU\}, \{UU, UD\}\},$$

(since the right hand side above is a σ-algebra). By the same reasoning, since X_2 is either 2, 8, or 32, and $X_2^{-1}(2) = \{DD\}$, $X_2^{-1}(8) = \{DU, UD\}$, and $X_2^{-1}(32) = \{UU\}$, we have

$$\begin{aligned}\mathcal{G}_2 = \sigma(X_2) &= \sigma(\{DD\}, \{DU, UD\}, \{UU\}) \\ &= \{\varnothing, \Omega, \{DD\}, \{DU, UD\}, \{UU\}, \{DD, DU, UD\}, \{DU, UD, UU\}, \\ &\quad \{DD, UU\}\}.\end{aligned}$$

Since $\{DD, DU\}, \{UU, UD\} \in \sigma(X_1)$ and $\{DU, UD\}, \{DD, UU\} \in \sigma(X_2)$, then by taking intersections we see that $\{DU\}, \{UD\} \in \sigma(X_1, X_2)$. Since also $\{DD\}, \{UU\} \in \sigma(X_2)$, we get

$$\mathcal{F}_2 = \sigma(X_0, X_1, X_2) = 2^\Omega = \mathcal{F}.$$

(d) By construction, $\mathbb{F}$ is a filtration on $(\Omega, \mathcal{F})$ (of course, this can also be checked directly). Since $\{DD, DU\} \in \mathcal{G}_1 \backslash \mathcal{G}_2$, then $\mathcal{G}_1 \subsetneq \mathcal{G}_2$, and hence $\mathbb{G}$ is not a filtration.

(e) By construction, X is adapted to $\mathbb{F}$. Also, X_k is of course $\mathcal{G}_k$-measurable for each $k = 0, 1, 2$, but $\mathbb{G}$ is not a filtration.

(f) The process X can be interpreted as the price of a stock that is worth 8 at time zero, and at each period changes by a factor of either 1/2 or 2.

The filtration $\mathbb{F}$ can be thought of as the *cumulative* information that the stock price evolution provides us with over time.

The collection $\mathbb{G}$ can be though of as the information we know by only observing the present stock price, but not the past stock prices.