Mathematical Foundations for Finance

Exercise sheet 1

The first three exercises of this sheet contain material which is ***fundamental*** *for the course and* ***assumed to be known****. Please hand in your solutions by 12:00 on Wednesday, October 4 via the course homepage.*

Exercise 1.1 Let Ω be a non-empty set.

(a) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Prove that $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a σ-algebra on Ω.

(b) Let $\mathcal{A}$ be a family of subsets of Ω. Show that there is a (clearly unique) minimal σ-algebra $\sigma(\mathcal{A})$ containing $\mathcal{A}$. Here minimality is with respect to inclusion: if $\mathcal{F}$ is a σ-algebra with $\mathcal{A} \subseteq \mathcal{F}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.

(c) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Show by example that $\mathcal{F}_1 \cup \mathcal{F}_2$ may fail to be a σ-algebra.
Hint: You can consider two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega := \{1, 2, 3\}$.

Exercise 1.2 Consider a probability space $(\Omega, \mathcal{F}, P)$. A σ-algebra $\mathcal{F}_0 \subseteq \mathcal{F}$ is said to be *P-trivial* if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{F}_0$. Prove that $\mathcal{F}_0$ is P-trivial if and only if every $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$ is P-a.s. constant.

Exercise 1.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Then, the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E[X\mathbb{1}_A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called the *conditional expectation of X given $\mathcal{G}$* and is denoted by $E[X \,|\, \mathcal{G}]$.
[*This is the formal definition of the conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.*]

(a) Show that if X is $\mathcal{G}$-measurable, then $E[X \,|\, \mathcal{G}] = X$ P-a.s.

(b) Show that $E[E[X \,|\, \mathcal{G}]] = E[X]$.

(c) Show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$ (that is, if $\mathcal{G}$ is P-trivial), then $E[X \,|\, \mathcal{G}] = E[X]$ P-a.s.

(d) Consider an integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and some constants $a, b \in \mathbb{R}$. Show that $E[aX + bY \,|\, \mathcal{G}] = aE[X \,|\, \mathcal{G}] + bE[Y \,|\, \mathcal{G}]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e., there exists a collection $(A_i)_{i=1,\dots,n}$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \dots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \dots, n$. Show that

$$E[X \,|\, \mathcal{G}] = \sum_{i=1}^{n} E[X \,|\, A_i] \, \mathbb{1}_{A_i} \; P\text{-a.s.}$$

This says that the conditional expectation of a random variable given a finitely generated σ-algebra is a *piecewise constant* function with the constants given by the elementary conditional expectations given the sets of the generating partition.

[*This is a very useful property when one conditions on a finitely generated σ-algebra, as for instance in the multinomial model.*]

Hint 1: Recall that $E[X \mid A_i] = E[X \mathbb{1}_{A_i}]/P[A_i]$ and try to write X as a sum of random variables each of which only takes non-zero values on a single A_i.
Hint 2: Check that any set $A \in \mathcal{G}$ has the form $\cup_{j\in J} A_j$ for some $J \subseteq \{1, \dots, n\}$.

Exercise 1.4 Let $(\Omega, \mathcal{F}, P)$ be the probability space with $\Omega = \{UU, UD, DD, DU\}$, $\mathcal{F} = 2^\Omega$, and P defined by $P[\omega] = 1/4$ for all $\omega \in \Omega$ (so P is the *uniform* probability measure on Ω). Consider the random variables $Y_1, Y_2 \colon \Omega \to \mathbb{R}$ that are given by $Y_1(UU) = Y_1(UD) = 2$, $Y_1(DD) = Y_1(DU) = 1/2$, $Y_2(UU) = Y_2(DU) = 2$, and $Y_2(DD) = Y_2(UD) = 1/2$. Define the process $X = (X_k)_{k=0,1,2}$ by $X_0 = 8$, and $X_k = X_0 \prod_{i=1}^k Y_i$ for $k = 1, 2$.

(a) Show that, for any continuous function $h : \mathbb{R} \to \mathbb{R}$, the composition $h(X_1)$ is $\sigma(X_1)$-measurable.

(b) Draw a tree to illustrate the possible evolutions of the process X from time 0 to time 2, and label the corresponding transition probabilities and probabilities.

(c) Write down the σ-algebras (i.e. give all their sets) defined by $\mathcal{F}_k = \sigma(X_i : 0 \leqslant i \leqslant k)$ and $\mathcal{G}_k = \sigma(X_k)$ for $k = 0, 1, 2$.

(d) Consider the collections of σ-algebras $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ and $\mathbb{G} = (\mathcal{G}_k)_{k=0,1,2}$. Do these form filtrations on $(\Omega, \mathcal{F})$? Why or why not?

(e) If they are indeed filtrations, is X adapted to $\mathbb{F}$ or $\mathbb{G}$?

(f) Give financial interpretations of X, $\mathbb{F}$ and $\mathbb{G}$.