Mathematical Foundations for Finance

Exercise Sheet 11

Exercise 11.1 Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale null at 0. We define the process

$$Z := \mathcal{E}(X) := e^{X-\frac{1}{2}[X]}.$$

(a) Show via Itô's formula that

$$Z_t = 1 + \int_0^t Z_s \mathrm{d}X_s,\ P\text{-a.s.},\ \text{for } t \geq 0. \tag{1}$$

Conclude that Z is a continuous local martingale if and only if X is a continuous local martingale.
Hint: You may compute Itô's formula for $f(x, y) := e^{x-\frac{1}{2}y}$.

(b) Show that $Z = \mathcal{E}(X)$ is the unique solution to (1).
Hint: You may compute Z'/Z *using Itô's formula, where* Z' *is another solution of Equation* (1).

(c) Let $Y = (Y_t)_{t\geq 0}$ be another continuous semimartingale null at 0. Prove *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}\left(X + Y + [X, Y]\right),\ P\text{-a.s.}$$

Hint: You may deduce this formula from the uniqueness proved at point (b).

Solution 11.1

(a) We apply Itô's formula to the C^2-function $f(x, y) := e^{x-\frac{1}{2}y}$ and the continuous semimartingale $(X_t, [X]_t)_{t\geq 0}$. We obtain that

$$\begin{aligned}
\mathrm{d}Z_t &= \mathrm{d}f(X_t, [X]_t) \\
&= \frac{\partial}{\partial x} f(X_t, [X]_t)\mathrm{d}X_t + \frac{\partial}{\partial y} f(X_t, [X]_t)\mathrm{d}[X]_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(X_t, [X]_t)\mathrm{d}[X]_t \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial y^2} f(X_t, [X]_t)\mathrm{d}\Big[[X]\Big]_t + \frac{\partial^2}{\partial x \partial y} f(X_t, [X]_t)\mathrm{d}\Big[X, [X]\Big]_t,\ P\text{-a.s.}.
\end{aligned}$$

However, since X is continuous and $[X]$ is continuous and of finite variation, we have that $\Big[[X]\Big] = 0$, P-a.s., and $\Big[X, [X]\Big] = 0$, P-a.s. Moreover, a direct computation shows that $\frac{\partial}{\partial y} f + \frac{1}{2}\frac{\partial^2}{\partial x^2} f = 0$ and $\frac{\partial}{\partial x} f = f$. We conclude that

$$\mathrm{d}Z_t = Z_t \mathrm{d}X_t,\ P\text{-a.s.},\ \text{or } Z_t = 1 + \int_0^t Z_s \mathrm{d}X_s, P\text{-a.s.}$$

As Z is a C^2-transformation of the continuous semimartingale $(X_t, [X]_t)_{t\geq 0}$, the process Z is always a continuous semimartingale (hence predictable and locally bounded). Therefore, $Z \in L^2_{\text{loc}}(M)$ for all continuous local martingales M. If X is a continuous local martingale, then we conclude that Z is a continuous local martingale.

Conversely, since Z is strictly positive by definition, X is given by

$$\mathrm{d}X_t = \frac{1}{Z_t}\mathrm{d}Z_t, P\text{-a.s., or } X_t = \int_0^t \frac{1}{Z_s}\mathrm{d}Z_s, P\text{-a.s.}$$

Therefore, if Z is a continuous local martingale, then X is a local martingale by the same reasoning as above.

(b) Let Z' be another process such that

$$\mathrm{d}Z'_t = Z'_t \mathrm{d}X_t, \;\; Z'_0 = 1, \; P\text{-a.s.}$$

Since Z' is necessarily a semimartingale, we can apply Itô's formula to the quotient $\frac{Z'}{Z} = f(Z', Z)$ with the function $f(x,y) = \frac{x}{y}$. A direct computation yields

$$\frac{\partial}{\partial x} f(x,y) = \frac{1}{y}, \quad \frac{\partial}{\partial y} f(x,y) = -\frac{x}{y^2},$$
$$\frac{\partial^2}{\partial x^2} f(x,y) = 0, \quad \frac{\partial^2}{\partial x \partial y} f(x,y) = -\frac{1}{y^2}, \quad \frac{\partial^2}{\partial y^2} f(x,y) = 2\frac{x}{y^3}.$$

Plugging these into Itô's formula and using that $\mathrm{d}Z_t = Z_t \mathrm{d}X_t$ and $\mathrm{d}Z'_t = Z'_t \mathrm{d}X_t$ gives that $\mathrm{d}[Z]_t = Z_t^2 \mathrm{d}[X]_t$, $\mathrm{d}[Z', Z]_t = Z'_t Z_t \mathrm{d}[X]_t$ which then yields

$$\begin{aligned}
\mathrm{d}\left(\frac{Z'_t}{Z_t}\right) &= \frac{1}{Z_t}\mathrm{d}Z'_t - \frac{Z'_t}{Z_t^2}\mathrm{d}Z_t - \frac{1}{Z_t^2}\mathrm{d}[Z', Z]_t + \frac{Z'_t}{Z_t^3}\mathrm{d}[Z]_t \\
&= \frac{Z'_t}{Z_t}\mathrm{d}X_t - \frac{Z'_t}{Z_t}\mathrm{d}X_t - \frac{Z'_t}{Z_t}\mathrm{d}[X]_t + \frac{Z'_t}{Z_t}\mathrm{d}[X]_t \\
&= 0, \; P\text{-a.s.}
\end{aligned}$$

Hence, we conclude that $\frac{Z'_t}{Z_t} = 1$, P-a.s., for all $t \geq 0$.

(c) The product rule implies that

$$\begin{aligned}
\mathrm{d}\Big(\mathcal{E}(X)\mathcal{E}(Y)\Big) &= \mathcal{E}(X)\mathrm{d}\mathcal{E}(Y) + \mathcal{E}(Y)\mathrm{d}\mathcal{E}(X) + \mathrm{d}[\mathcal{E}(X), \mathcal{E}(Y)] \\
&= \mathcal{E}(X)\mathcal{E}(Y)\mathrm{d}Y + \mathcal{E}(Y)\mathcal{E}(X)\mathrm{d}X + \mathcal{E}(X)\mathcal{E}(Y)\mathrm{d}[X,Y] \\
&= \mathcal{E}(X)\mathcal{E}(Y)\mathrm{d}\left(X + Y + [X,Y]\right).
\end{aligned}$$

By uniqueness of the solution to $\mathrm{d}Z = Z\mathrm{d}X$ for X replaced by $X + Y + [X,Y]$, we conclude that

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X,Y]).$$

Exercise 11.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. Consider two independent Brownian motions $W^1 = (W_t^1)_{t\in[0,T]}$ and $W^2 = (W_t^2)_{t\in[0,T]}$, and let $\widetilde{S}^1 = (\widetilde{S}_t^1)_{t\in[0,T]}$ and $\widetilde{S}^2 = (\widetilde{S}_t^2)_{t\in[0,T]}$ be two processes with the dynamics

$$\begin{aligned} \mathrm{d}\widetilde{S}_t^1 &= \widetilde{S}_t^1 \left(\mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}B_t^1\right),\ P\text{-a.s.},\ \widetilde{S}_0^1 > 0, \\ \mathrm{d}\widetilde{S}_t^2 &= \widetilde{S}_t^2 \left(\mu_2 \mathrm{d}t + \sigma_2 \mathrm{d}B_t^2\right),\ P\text{-a.s.},\ \widetilde{S}_0^2 > 0, \end{aligned}$$

where $B^1 := W^1$ and $B^2 := \alpha W^1 + \sqrt{1-\alpha^2} W^2$, for some $\alpha \in (-1,1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

(a) Find the SDEs describing the dynamics of $X^1 := \frac{\widetilde{S}^2}{\widetilde{S}^1}$ and $X^2 := \frac{\widetilde{S}^1}{\widetilde{S}^2}$, expressed in terms of B^1 and B^2.

(b) Fix some $\beta_1, \beta_2 \in \mathbb{R}$, and define the continuous local martingale

$$L^{(\beta_1,\beta_2)} := \beta_1 W^1 + \beta_2 W^2.$$

Show that the stochastic exponential $Z^{(\beta_1,\beta_2)} := \mathcal{E}(L^{(\beta_1,\beta_2)})$ is a true martingale on $[0,T]$.
Hint: You may use the independence of W^1 and W^2 and Proposition IV.2.3 in the lecture notes.

(c) Fix some $\beta_1, \beta_2 \in \mathbb{R}$, and define the probability measure $Q^{(\beta_1,\beta_2)}$, which is equivalent to P on $\mathcal{F}_T$, by

$$\mathrm{d}Q^{(\beta_1,\beta_2)} = Z_T^{(\beta_1,\beta_2)} \mathrm{d}P.$$

Show that $Z^{(\beta_1,\beta_2)}$ is the density process of $Q^{(\beta_1,\beta_2)}$ with respect to P on $[0,T]$. Using Girsanov's theorem, prove that the two processes $\widetilde{W}_t^1 := W_t^1 - \beta_1 t$ and $\widetilde{W}_t^2 := W_t^2 - \beta_2 t$, for $t \in [0,T]$, are local $Q^{(\beta_1,\beta_2)}$-martingales. Conclude that

$$\widetilde{B}^1 := \widetilde{W}^1 \text{ and } \widetilde{B}_t^2 := B_t^2 - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t, \text{ for } t \in [0,T],$$

are local $Q^{(\beta_1,\beta_2)}$-martingales as well.

(d) What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the processes X^1 and X^2 $Q^{(\beta_1,\beta_2)}$-martingales? Can they be martingales simultaneously under the same measure?
Hint: You may rewrite the SDEs describing the dynamics of X^1 and X^2 in terms of $\widetilde{W}^1$ and $\widetilde{W}^2$, and use the fact (without proving it) that $\widetilde{W}^1$ and $\widetilde{W}^2$ are independent Brownian motions under $Q^{(\beta_1,\beta_2)}$ (the reasoning is analogous to point (b)).

Solution 11.2

(a) Take $i \neq j$, where $i, j \in \{1, 2\}$. By Itô's formula, we get

$$\begin{aligned} \mathrm{d}X^i = \mathrm{d}\left(\frac{\widetilde{S}^j}{\widetilde{S}^i}\right) &= \frac{1}{\widetilde{S}^i}\mathrm{d}\widetilde{S}^j - \frac{\widetilde{S}^j}{\big(\widetilde{S}^i\big)^2}\mathrm{d}\widetilde{S}^i - \frac{1}{\big(\widetilde{S}^i\big)^2}\mathrm{d}[\widetilde{S}^i, \widetilde{S}^j] + \frac{\widetilde{S}^j}{\big(\widetilde{S}^i\big)^3}\mathrm{d}[\widetilde{S}^i] \\ &= X^i \left((\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j)\mathrm{d}t + \sigma_j \mathrm{d}B^j - \sigma_i \mathrm{d}B^i\right),\ P\text{-a.s.} \end{aligned}$$

(b) Fix some $\beta_1, \beta_2 \in \mathbb{R}$. Then, $L^{(\beta_1,\beta_2)}$ is clearly a martingale, whose quadratic variation satisfies, for all $t \in [0, T]$,

$$\left[L^{(\beta_1,\beta_2)}\right]_t = [\beta_1 W^1 + \beta_2 W^2]_t = \beta_1^2 t + \beta_2^2 t,\ P\text{-a.s.},$$

where we have used that $[W^1, W^2] = 0$, P-a.s. Moreover, by independence of W^1 and W^2 and Proposition IV.2.3 in the lecture notes, we have

$$\begin{aligned} E\left[\frac{Z_t^{(\beta_1,\beta_2)}}{Z_s^{(\beta_1,\beta_2)}}\,\middle|\,\mathcal{F}_s\right] &= E\left[\frac{e^{\beta_1 W_t^1 + \beta_2 W_t^2 - \frac{1}{2}(\beta_1^2+\beta_2^2)t}}{e^{\beta_1 W_s^1 + \beta_2 W_s^2 - \frac{1}{2}(\beta_1^2+\beta_2^2)s}}\,\middle|\,\mathcal{F}_s\right] \\ &= E\left[e^{\beta_1(W_t^1 - W_s^1) + \beta_2(W_t^2 - W_s^2) - \frac{1}{2}(\beta_1^2+\beta_2^2)(t-s)}\,\middle|\,\mathcal{F}_s\right] \\ &= e^{-\frac{1}{2}(\beta_1^2+\beta_2^2)(t-s)} E\left[e^{\beta_1(W_t^1 - W_s^1) + \beta_2(W_t^2 - W_s^2)}\,\middle|\,\mathcal{F}_s\right] \\ &= e^{-\frac{1}{2}\beta_1^2(t-s)} E\left[e^{\beta_1(W_t^1 - W_s^1)}\right] e^{-\frac{1}{2}\beta_2^2(t-s)} E\left[e^{\beta_2(W_t^2 - W_s^2)}\right] \\ &= 1,\ P\text{-a.s., for } s, t \in [0, T] \text{ with } s \leq t, \end{aligned}$$

so $Z^{(\beta_1,\beta_2)}$ has the martingale property. Adaptedness is clear and the integrability follows from the fact that $Z_t^{(\beta_1,\beta_2)}$ is a log-normally distributed random variable for all $t \in [0, T]$, and we know that all moments of log-normal distributions are finite. Therefore, $Z_t^{(\beta_1,\beta_2)}$ is a martingale.

(c) We prove that $Z_t^{(\beta_1,\beta_2)} = \tilde{Z}_t^{(\beta_1,\beta_2)}$, P-a.s., for any $t \in [0, T]$, where $\tilde{Z}^{(\beta_1,\beta_2)}$ denotes the density process of $Q^{(\beta_1,\beta_2)}$ with respect to P on $[0, T]$. Let us fix $t \in [0, T]$, and some $A \in \mathcal{F}_t$. It holds that

$$\begin{aligned} E\Big[1_A \tilde{Z}_t^{(\beta_1,\beta_2)}\Big] &= E^P\Big[1_A \tilde{Z}_t^{(\beta_1,\beta_2)}\Big] \\ &= E^{P|\mathcal{F}_t}\Big[1_A \tilde{Z}_t^{(\beta_1,\beta_2)}\Big] \\ &= E^{Q^{(\beta_1,\beta_2)}|\mathcal{F}_t}[1_A] \\ &= E^{Q^{(\beta_1,\beta_2)}}[1_A] \\ &= E^P\left[1_A \frac{\mathrm{d}Q^{(\beta_1,\beta_2)}}{\mathrm{d}P}\right] \\ &= E^P\Big[1_A Z_T^{(\beta_1,\beta_2)}\Big] \\ &= E^P\Big[1_A\ E^P\Big[Z_T^{(\beta_1,\beta_2)} | \mathcal{F}_t\Big]\Big]. \end{aligned}$$

Using the martingale property of $Z^{(\beta_1,\beta_2)}$, we deduce that

$$E\Big[1_A \tilde{Z}_t^{(\beta_1,\beta_2)}\Big] = E\Big[1_A\ Z_t^{(\beta_1,\beta_2)}\Big],$$

and we conclude that $\tilde{Z}_t^{(\beta_1,\beta_2)} = Z_t^{(\beta_1,\beta_2)}$, P-a.s., by the arbitrariness of A.

By Girsanov's theorem in the form of Theorem VI.2.3 in the lecture notes, we know that

$$W^1 - [L^{(\beta_1,\beta_2)}, W^1] \text{ and } W^2 - [L^{(\beta_1,\beta_2)}, W^2]$$

are local $Q^{(\beta_1,\beta_2)}$-martingales. Thus, it suffices to show that for all $t \in [0,T]$, we have

$$\Big[L^{(\beta_1,\beta_2)}, W^1\Big]_t = \beta_1 t,\ P\text{-a.s., and } [L^{(\beta_1,\beta_2)}, W^2]_t = \beta_2 t,\ P\text{-a.s.}$$

But this follows immediately from the independence of W^1 and W^2 and the definition of $L^{(\beta_1,\beta_2)}$.

To conclude, we simply write the definition of the corresponding process $\widetilde{B}^2$ to get

$$\begin{aligned}\widetilde{B}_t^2 &:= B_t^2 - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t := \alpha(W_t^1 - \beta_1 t) + \sqrt{1-\alpha^2}(W_t^2 - \beta_2 t) \\ &= \alpha\widetilde{W}_t^1 + \sqrt{1-\alpha^2}\widetilde{W}_t^2,\ \text{for } t \in [0,T],\end{aligned} \tag{2}$$

which is a linear combination of local $Q^{(\beta_1,\beta_2)}$-martingales.

(d) First, we note that X^1 and X^2 still satisfy the same SDEs under $Q^{(\beta_1,\beta_2)}$ with the only difference that B^1 and B^2 are in general no longer Brownian motions under $Q^{(\beta_1,\beta_2)}$. Using that $\widetilde{B}^1$ and $\widetilde{B}^2$ are local martingales under $Q^{(\beta_1,\beta_2)}$, we get by (a) that

$$\begin{aligned}\mathrm{d}X^i &= X^i\left((\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j)\mathrm{d}t + \sigma_j \mathrm{d}(\widetilde{B}^j + \gamma_j t) - \sigma_i \mathrm{d}(\widetilde{B}^i + \gamma_i t)\right) \\ &= X^i\left((\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j + \sigma_j\gamma_j - \sigma_i\gamma_i)\mathrm{d}t + \sigma_j \mathrm{d}\widetilde{B}^j - \sigma_i \mathrm{d}\widetilde{B}^i\right),\end{aligned} \tag{3}$$

where $\gamma_1 := \beta_1$ and $\gamma_2 := \alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2$. Next, X^i is a local $Q^{(\beta_1,\beta_2)}$-martingale if and only if the drift component in (3) vanishes, i.e.,

$$\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j + \sigma_j\gamma_j - \sigma_i\gamma_i = 0. \tag{4}$$

Now, we express the local martingale components in terms of $\widetilde{W}^1$ and $\widetilde{W}^2$ (see Equation (2)),

$$\begin{cases} \sigma_1 \mathrm{d}\widetilde{B}^1 - \sigma_2 \mathrm{d}\widetilde{B}^2 = (\sigma_1 - \sigma_2\alpha)\mathrm{d}\widetilde{W}^1 - \sigma_2\sqrt{1-\alpha^2}\mathrm{d}\widetilde{W}^2, \\ \sigma_2 \mathrm{d}\widetilde{B}^2 - \sigma_1 \mathrm{d}\widetilde{B}^1 = \sigma_2\sqrt{1-\alpha^2}\mathrm{d}\widetilde{W}^2 - (\sigma_1 - \sigma_2\alpha)\mathrm{d}\widetilde{W}^1. \end{cases}$$

Since $\widetilde{W}^1$ and $\widetilde{W}^2$ are independent Brownian motions under $Q^{(\beta_1,\beta_2)}$, we may argue analogously to (b) that X^1 and X^2 are true $Q^{(\beta_1,\beta_2)}$-martingales provided that (4) holds.

Finally, either X^1 or X^2 can be a martingale but not both simultaneously. In fact, because $X^2 = 1/X^1$ and $\mathbb{R}_+ \ni x \mapsto 1/x$ is a strictly convex function, Jensen's inequality gives $\tilde{P}\Big(E^{\tilde{P}}[X_t^2|\mathcal{F}_s] > 1/E^{\tilde{P}}[X_t^1|\mathcal{F}_s]\Big) > 0$, for $s,t \in [0,T]$ with $s \leq t$, for any probability $\tilde{P}$ such that X^1 is a true $\tilde{P}$-martingale.

Exercise 11.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. Let $M = (M_t)_{t\in[0,T]}$ be a local martingale and $W = (W_t)_{t\in[0,T]}$ a Brownian motion.

(a) Let $H = (H_t)_{t\in[0,T]}$ be in $L^2(M)$. Compute $E\left[\int_0^T H_s \mathrm{d}M_s\right]$ and $\mathrm{Var}\left[\int_0^T H_s \mathrm{d}M_s\right]$. How do the expressions look like when $M := W$?

(b) By finding a counterexample, show that the random variable $\int_0^T H_s \mathrm{d}W_s$ is not normally distributed for any arbitrary continuous process $H \in L^2(W)$.
Hint: You may use the example at page 106 of the lecture notes.

Solution 11.3

(a) Since $H \in L^2(M)$, we know that $(H \bullet M) \in \mathcal{M}_0^2$, i.e. $(H \bullet M)$ is a square-integrable RCLL $(P, \mathbb{F})$-martingale null at 0 with $\sup_{t\geq 0} E\left[(H \bullet M)_t^2\right] < \infty$. In particular, both $E\left[(H \bullet M)_T\right]$ and $\mathrm{Var}\left[(H \bullet M)_T\right]$ are finite for all $T \geq 0$. We must therefore have that

$$E\left[\int_0^T H_s \mathrm{d}M_s\right] = E\left[E\left[\int_0^T H_s \mathrm{d}M_s \,\middle|\, \mathcal{F}_0\right]\right] = E\left[(H \bullet M)_0\right] = 0.$$

For the variance, we compute

$$\mathrm{Var}\left[\int_0^T H_s \mathrm{d}M_s\right] = E\left[\left(\int_0^T H_s \mathrm{d}M_s\right)^2\right] = E\left[\int_0^T H_s^2 \mathrm{d}[M]_s\right],$$

where the first equality uses that $E\left[(H \bullet M)_T\right] = 0$ and the second the isometry property of the stochastic integral with respect to a local martingale.

In the particular case of a Brownian motion, i.e. when $M = W$, no further simplification is needed for $E\left[(H \bullet W)_T\right]$. As for $\mathrm{Var}\left[(H \bullet W)_T\right]$, we have that

$$E\left[\int_0^T H_s^2 \mathrm{d}[W]_s\right] = E\left[\int_0^T H_s^2 \mathrm{d}s\right] = \int_0^T E\left[H_s^2\right] \mathrm{d}s,$$

where the last equality uses Fubini's theorem.

(b) From the example at page 106 of the lecture notes, we have that

$$\int_0^T W_s \mathrm{d}W_s = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

Since W_T^2 only takes positive values, it is clear that $\frac{1}{2}W_T^2 - \frac{1}{2}T$ is bounded from below for every $T \geq 0$, so it cannot be normally distributed. More specifically,

$$\frac{1}{2}W_T^2 - \frac{1}{2}T \overset{d}{=} \frac{1}{2}\left(\sqrt{T}W_1\right)^2 - \frac{1}{2}T,$$

and since $W_1 \sim \mathcal{N}(0,1)$, we have that $W_1^2 \sim \chi_1^2$, so the distribution of $\int_0^T W_s \mathrm{d}W_s$ is just a simple affine transformation of a chi-squared distribution with one degree of freedom.