Mathematical Foundations for Finance

Exercise Sheet 11

Please hand in your solutions by 12:00 on Wednesday, December 13 via the course homepage.

Exercise 11.1 Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale null at 0. We define the process

$$Z := \mathcal{E}(X) := e^{X - \frac{1}{2}[X]}.$$

(a) Show via Itô's formula that

$$Z_t = 1 + \int_0^t Z_s \mathrm{d}X_s,\ P\text{-a.s., for } t \geq 0. \tag{1}$$

Conclude that Z is a continuous local martingale if and only if X is a continuous local martingale.
Hint: You may compute Itô's formula for $f(x,y) := e^{x - \frac{1}{2}y}$.

(b) Show that $Z = \mathcal{E}(X)$ is the unique solution to (1).
Hint: You may compute Z'/Z *using Itô's formula, where* Z' *is another solution of Equation* (1).

(c) Let $Y = (Y_t)_{t\geq 0}$ be another continuous semimartingale null at 0. Prove *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}\left(X + Y + [X,Y]\right),\ P\text{-a.s.}$$

Hint: You may deduce this formula from the uniqueness proved at point (b).

Exercise 11.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. Consider two independent Brownian motions $W^1 = (W^1_t)_{t\in[0,T]}$ and $W^2 = (W^2_t)_{t\in[0,T]}$, and let $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ and $\widetilde{S}^2 = (\widetilde{S}^2_t)_{t\in[0,T]}$ be two processes with the dynamics

$$\mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t \left(\mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}B^1_t\right),\ P\text{-a.s., } \widetilde{S}^1_0 > 0,$$
$$\mathrm{d}\widetilde{S}^2_t = \widetilde{S}^2_t \left(\mu_2 \mathrm{d}t + \sigma_2 \mathrm{d}B^2_t\right),\ P\text{-a.s., } \widetilde{S}^2_0 > 0,$$

where $B^1 := W^1$ and $B^2 := \alpha W^1 + \sqrt{1-\alpha^2}W^2$, for some $\alpha \in (-1,1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

(a) Find the SDEs satisfied by $X^1 := \frac{\widetilde{S}^2}{\widetilde{S}^1}$ and $X^2 := \frac{\widetilde{S}^1}{\widetilde{S}^2}$, expressed in terms of B^1 and B^2.

(b) Fix some $\beta_1, \beta_2 \in \mathbb{R}$, and define the continuous local martingale

$$L^{(\beta_1,\beta_2)} := \beta_1 W^1 + \beta_2 W^2.$$

Show that the stochastic exponential $Z^{(\beta_1,\beta_2)} := \mathcal{E}(L^{(\beta_1,\beta_2)})$ is a true martingale on $[0,T]$.
Hint: You may use the independence of W^1 and W^2 and Proposition IV.2.3 in the lecture notes.

(c) Fix some $\beta_1, \beta_2 \in \mathbb{R}$, and define the probability measure $Q^{(\beta_1,\beta_2)}$, which is equivalent to P on $\mathcal{F}_T$, by

$$\mathrm{d}Q^{(\beta_1,\beta_2)} = Z_T^{(\beta_1,\beta_2)}\mathrm{d}P.$$

Show that $Z^{(\beta_1,\beta_2)}$ is the density process of $Q^{(\beta_1,\beta_2)}$ with respect to P on $[0,T]$. Using Girsanov's theorem, prove that the two processes $\widetilde{W}_t^1 := W_t^1 - \beta_1 t$ and $\widetilde{W}_t^2 := W_t^2 - \beta_2 t$, for $t \in [0,T]$, are local $Q^{(\beta_1,\beta_2)}$-martingales. Conclude that

$$\widetilde{B}^1 := \widetilde{W}^1 \text{ and } \widetilde{B}_t^2 := B_t^2 - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t, \text{ for } t \in [0,T],$$

are local $Q^{(\beta_1,\beta_2)}$-martingales as well.

(d) What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the processes X^1 and X^2 $Q^{(\beta_1,\beta_2)}$-martingales? Can they be martingales simultaneously under the same measure?
Hint: You may rewrite the SDEs satisfied by X^1 and X^2 in terms of $\widetilde{W}^1$ and $\widetilde{W}^2$, and use the fact (without proving it) that $\widetilde{W}^1$ and $\widetilde{W}^2$ are independent Brownian motions under $Q^{(\beta_1,\beta_2)}$ (the reasoning is analogous to point (b)).

Exercise 11.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. Let $M = (M_t)_{t\in[0,T]}$ be a local martingale and $W = (W_t)_{t\in[0,T]}$ a Brownian motion.

(a) Let $H = (H_t)_{t\in[0,T]}$ be in $L^2(M)$. Compute $E\left[\int_0^T H_s \mathrm{d}M_s\right]$ and $\mathrm{Var}\left[\int_0^T H_s \mathrm{d}M_s\right]$. How do the expressions look like when $M := W$?

(b) By finding a counterexample, show that the random variable $\int_0^T H_s \mathrm{d}W_s$ is not normally distributed for any arbitrary continuous process $H \in L^2(W)$.
Hint: You may use the example at page 106 of the lecture notes.