Mathematical Foundations for Finance

Exercise Sheet 3

Exercise 3.1 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$.

(a) Let X be a martingale. Show that for any bounded and convex function $f\colon \mathbb{R} \to \mathbb{R}$, the process $f(X) = (f(X_k))_{k\in\mathbb{N}_0}$ is a submartingale.

Could we replace the request of f being bounded with a more general condition?

Hint: You may use that finite-valued convex functions are continuous.

(b) Let X be a submartingale, and let $\vartheta = (\vartheta_k)_{k\in\mathbb{N}_0}$ be a bounded, nonnegative and predictable process. Show that the stochastic integral process $\vartheta \bullet X$, defined by

$$\vartheta \bullet X_k = \sum_{j=1}^{k} \vartheta_j \Delta X_j = \sum_{j=1}^{k} \vartheta_j (X_j - X_{j-1}),$$

is a submartingale.

Conclude that $E[\vartheta \bullet X_k] \geqslant 0$ for all $k \in \mathbb{N}_0$.

(c) Let X be a submartingale and let τ be a stopping time. Show that the stopped process $X^\tau = (X^\tau_k)_{k\in\mathbb{N}_0}$ defined by $X^\tau_k = X_{k\wedge\tau}$ is a submartingale.

Solution 3.1

(a) The process $f(X)$ is integrable because f is bounded. Since X is adapted (because it is a martingale) and f is continuous (since it is finite-valued and convex), it follows that $f(X)$ is adapted. It remains to show the submartingale inequality. For $0 \leqslant m < n$, we write

$$E[f(X_n) \mid \mathcal{F}_m] \geqslant f(E[X_n \mid \mathcal{F}_m]) = f(X_m),$$

where the first step used the (conditional) Jensen's inequality, and the second step the martingale property. This concludes the proof.

A look at the proof shows that if we replace the condition "f is bounded" by "$f(X)$ is integrable", the result still holds.

(b) Since ϑ is predictable and X is adapted, then $\vartheta_j(X_j - X_{j-1})$ is $\mathcal{F}_j$-measurable for all $j \in \mathbb{N}$. It follows that $\vartheta \bullet X_k$ is $\mathcal{F}_k$-measurable, so that $\vartheta \bullet X$ is adapted. Also, since ϑ is bounded and X is integrable, we have that $\vartheta \bullet X$ is integrable.

It remains to establish the submartingale inequality. Note that it suffices to show

$$E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] \geqslant 0, \ \forall k \in \mathbb{N}_0.$$

To this end, we write

$$\begin{aligned} E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] &= E[\vartheta_{k+1}(X_{k+1} - X_k) \mid \mathcal{F}_k] \\ &= \vartheta_{k+1} E[X_{k+1} - X_k \mid \mathcal{F}_k], \end{aligned}$$

where in the last step we used that ϑ_{k+1} is $\mathcal{F}_k$-measurable and bounded. Since X is a submartingale, then $E[X_{k+1} - X_k \mid \mathcal{F}_k] \geqslant 0$. Since also ϑ_{k+1} is nonnegative by assumption, we have

$$E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] \geqslant 0,$$

as required.

Since $\vartheta \bullet X$ is a submartingale null at zero, we have for all $k \in \mathbb{N}_0$ that

$$E[\vartheta \bullet X_k] = E\Big[E[\vartheta \bullet X_k \mid \mathcal{F}_0]\Big] \geqslant E[\vartheta \bullet X_0] = 0.$$

(c) For $k \in \mathbb{N}_0$, we have

$$X_k^\tau = X_{k \wedge \tau} = X_0 + \sum_{j=1}^{k \wedge \tau} (X_j - X_{j-1}) = X_0 + \sum_{j=1}^{k} \mathbb{1}_{\{\tau \geqslant j\}} (X_j - X_{j-1}).$$

So if we set $\vartheta = (\vartheta_k)_{k \in \mathbb{N}}$ with $\vartheta_k := \mathbb{1}_{\{\tau \geqslant k\}}$, then

$$X_k^\tau = X_0 + \vartheta \bullet X_k, \ \forall k \in \mathbb{N}_0.$$

Since τ is a stopping time, then ϑ is a predictable process. Since ϑ is also bounded and nonnegative, and X is a submartingale, we may apply part (b) to conclude that $\vartheta \bullet X$ is a submartingale. Also, note that because X_0 is $\mathcal{F}_0$-measurable and integrable, then the process $(X_0)_{k \in \mathbb{N}_0}$ is a submartingale (in fact a martingale). Since the sum of two submartingales is a submartingale, we can conclude that X^τ is a submartingale, as required.

Exercise 3.2 Fix $u > d > -1$ and a finite time horizon $T \in \mathbb{N}$. Let $Y_1, \dots, Y_T$ be i.i.d. random variables with distribution given by

$$P[Y_k = 1 + u] = p, \quad P[Y_k = 1 + d] = 1 - p,$$

where $p \in (0,1)$ is fixed. Also, fix $r > -1$, and let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with the price processes of the assets in our market given by $\widetilde{S}_0^1 = 1$ and

$$\begin{aligned} \widetilde{S}_k^0 &= (1+r)^k, \ k = 0, \dots, T, \\ \frac{\widetilde{S}_k^1}{\widetilde{S}_{k-1}^1} &= Y_k, \qquad k = 1, \dots, T. \end{aligned}$$

(a) By constructing an arbitrage opportunity, show that the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits arbitrage if $r \leqslant d$.

(b) Show that the same holds if $r \geqslant u$.

Solution 3.2

(a) If $r \leq d$, the stock grows at least as fast as the bank account, and possibly faster since $u > d$. Formally, we have that for all $k = 1, \ldots, T$,

$$Y_k \geqslant 1 + r, \quad P[Y_k > 1 + r] > 0,$$

and therefore

$$S_k^1 \geqslant S_{k-1}^1, \quad P[S_k^1 > S_{k-1}^1] > 0. \tag{1}$$

We therefore obtain an arbitrage opportunity as follows: at time 0, borrow money from the bank account to buy, say, one share of the stock, and hold it until time T. With probability 1, we will be able to completely repay the debt (by using the value of the stock), and with strictly positive probability, we will even have some money left over.

Formally, we are considering the self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$, where $V_0 = 0$ and $\vartheta = (\vartheta_k)_{k=0,\ldots,T}$ is given by $\vartheta_0 = 0$ and $\vartheta_k = 1$ for $k = 1, \ldots, T$. Since ϑ is deterministic, it is of course predictable, and thus φ is a self-financing strategy by Proposition 2.3, and

$$V_k(\varphi) = V_0 + G_k(\vartheta) = \sum_{j=1}^{k} \vartheta_j \Delta S_j^1 = \sum_{j=1}^{k} \Delta S_j^1 = S_k^1 - S_0^1 = S_k^1 - 1.$$

Finally, (1) shows that

$$P[V_T(\varphi) \geqslant 0] = P[S_T^1 \geqslant 1] = 1 \quad \text{and} \quad P[V_T(\varphi) > 0] > 0.$$

Hence, φ is an arbitrage opportunity, as required.

(b) If $r \geqslant u$, then we consider the opposite strategy of part (a). At time zero, we sell short 1 share of the stock, and put the profits in the bank account. We hold our money in the bank until time T, and then buy the stock back to repay our debts. With probability 1, we will be able to repay our debt, in with strictly positive probability, we will have some money left over.

The explicit mathematical formulation of the arbitrage opportunity is very similar to part (a), and is therefore omitted.

Exercise 3.3 Let $\vartheta = (\vartheta_k)_{k=0,\ldots,T}$ be a predictable process with $\vartheta_0 = 0$, and let $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ be the corresponding self-financing strategy with initial capital 0.

(a) Show that if φ is not admissible, then there exists some $k \in \{0, \dots, T\}$ with $P[G_k(\vartheta) < 0] > 0$.

(b) Suppose that φ also satisfies $V_T(\varphi) \geqslant 0$ P-a.s., and $P[V_T(\varphi) > 0] > 0$. Construct a modification φ' of φ so that the corresponding self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$ is 0-admissible and satisfies $V_T(\varphi') \geqslant 0$ P-a.s., and $P[V_T(\varphi') > 0] > 0$.

Solution 3.3

(a) Suppose for contradiction that $P[G_k(\vartheta) \geqslant 0] = 1$ for all $k = 0, \dots, T$. Then since $V_0 = 0$ and φ is self-financing, for any $k = 0, \dots, T$, it holds that

$$V_k(\varphi) = G_k(\vartheta) \geqslant 0 \ P\text{-a.s.},$$

so that φ is 0-admissible. This contradicts our assumption, and thus completes the proof.

(b) If φ is 0-admissible, we can take $\varphi' := \varphi$. So assume φ is not 0-admissible. We construct a new strategy $\phi' \mathrel{\widehat{=}} (0, \vartheta')$ as follows. Define

$$k_0 := \max\{k : P[G_k(\vartheta) < 0] > 0\}$$

to be the last time that $G(\vartheta)$ is strictly negative with some positive probability (which exists by assumption), and let

$$A := \{G_{k_0}(\vartheta) < 0\}$$

be the set on which this happens. Note that $k_0 \geqslant 1$ because $G_0(\vartheta) = 0$, and $k_0 \leqslant T - 1$ because $G_T(\vartheta) = V_T(\varphi) \geqslant 0$ P-a.s. by assumption. We then define ϑ' so that the corresponding self-financing strategy φ' is to wait until time k_0, and then to follow φ on A, i.e.,

$$\vartheta'_k := \begin{cases} 0 & \text{if } k \leqslant k_0, \\ \vartheta_k \mathbb{1}_A & \text{if } k > k_0. \end{cases}$$

Since $A = \{G_{k_0}(\vartheta) < 0\} \in \mathcal{F}_{k_0}$ and ϑ is predictable, the product $\vartheta_k \mathbb{1}_A$ is $\mathcal{F}_{k-1}$-measurable for all $k > k_0$. Since $\vartheta'_k = 0$ for $k \leqslant k_0$, it follows that ϑ' is predictable, and thus indeed induces a self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$. Next, we compute

$$V_k(\varphi') = G_k(\vartheta') = \sum_{j=1}^{k} (\vartheta'_j)^T \Delta S_j^1 = \begin{cases} 0 & \text{if } k \leqslant k_0, \\ (G_k(\vartheta) - G_{k_0}(\vartheta)) \mathbb{1}_A & \text{if } k > k_0. \end{cases}$$

Since $G_{k_0}(\vartheta) < 0$ on A and $G_k(\vartheta) \geqslant 0$ P-a.s. for all $k > k_0$ by the maximality of k_0, it follows that $V_k(\varphi') \geqslant 0$ P-a.s. for any $k = 0, \dots, T$. Thus, φ' is 0-admissible.

It now remains to check $P[V_T(\varphi') > 0] > 0$. To this end, we write

$$V_T(\varphi') = \Big(G_T(\vartheta) - G_{k_0}(\vartheta)\Big)\mathbb{1}_A = \Big(V_T(\varphi) - G_{k_0}(\vartheta)\Big)\mathbb{1}_A \geqslant -G_{k_0}(\vartheta)\mathbb{1}_A,$$

where we used that $V_T(\varphi) \geqslant 0$ P-a.s. Since $G_{k_0}(\vartheta) < 0$ on A, we have

$$P[V_T(\varphi') > 0] \geqslant P[-G_{k_0}(\vartheta)\mathbb{1}_A > 0] = P[A] > 0,$$

as required.