Coordinator
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Mathematical Foundations for Finance

Exercise Sheet 3

Please hand in your solutions by 12:00 on Wednesday, October 18 via the course homepage.

Exercise 3.1 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$.

(a) Let X be a martingale. Show that for any bounded and convex function $f : \mathbb{R} \to \mathbb{R}$, the process $f(X) = (f(X_k))_{k \in \mathbb{N}_0}$ is a submartingale.

Could we replace the request of f being bounded with a more general condition?

Hint: You may use that finite-valued convex functions are continuous.

(b) Let X be a submartingale, and let $\vartheta = (\vartheta_k)_{k \in \mathbb{N}_0}$ be a bounded, nonnegative and predictable process. Show that the stochastic integral process $\vartheta \bullet X$, defined by

$$\vartheta \bullet X_k = \sum_{j=1}^{k} \vartheta_j \Delta X_j = \sum_{j=1}^{k} \vartheta_j (X_j - X_{j-1}),$$

is a submartingale.

Conclude that $E[\vartheta \bullet X_k] \geqslant 0$ for all $k \in \mathbb{N}_0$.

(c) Let X be a submartingale and let τ be a stopping time. Show that the stopped process $X^\tau = (X^\tau_k)_{k \in \mathbb{N}_0}$ defined by $X^\tau_k = X_{k \wedge \tau}$ is a submartingale.

Exercise 3.2 Fix $u > d > -1$ and a finite time horizon $T \in \mathbb{N}$. Let $Y_1, \dots, Y_T$ be i.i.d. random variables with distribution given by

$$P[Y_k = 1 + u] = p, \qquad P[Y_k = 1 + d] = 1 - p,$$

where $p \in (0, 1)$ is fixed. Also, fix $r > -1$, and let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with the price processes of the assets in our market given by $\widetilde{S}^1_0 = 1$ and

$$\widetilde{S}^0_k = (1 + r)^k, \; k = 0, \dots, T,$$
$$\frac{\widetilde{S}^1_k}{\widetilde{S}^1_{k-1}} = Y_k, \qquad k = 1, \dots, T.$$

(a) By constructing an arbitrage opportunity, show that the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits arbitrage if $r \leqslant d$.

(b) Show that the same holds if $r \geqslant u$.

Exercise 3.3 Let $\vartheta = (\vartheta_k)_{k=0,\dots,T}$ be a predictable process with $\vartheta_0 = 0$, and let $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ be the corresponding self-financing strategy with initial capital 0.

(a) Show that if φ is not admissible, then there exists some $k \in \{0, \dots, T\}$ with $P[G_k(\vartheta) < 0] > 0$.

(b) Suppose that φ also satisfies $V_T(\varphi) \geqslant 0$ P-a.s., and $P[V_T(\varphi) > 0] > 0$. Construct a modification φ' of φ so that the corresponding self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$ is 0-admissible and satisfies $V_T(\varphi') \geqslant 0$ P-a.s., and $P[V_T(\varphi') > 0] > 0$.