Mathematical Foundations for Finance

Exercise Sheet 4

Exercise 4.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. For any stopping time τ, we define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \colon A \cap \{\tau \le k\} \in \mathcal{F}_k \text{ for all } k = 0, 1, \dots, T\}.$$

(a) Show that $\mathcal{F}_\tau$ is a σ-algebra.

(b) Suppose σ, τ are two $\mathbb{F}$-stopping times with $\sigma(\omega) \le \tau(\omega)$ for all $\omega \in \Omega$. Show that $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. Conclude that if $\tau \equiv k_0$ for a fixed $k_0 \in \{0, 1, \dots, T\}$, then we have $\mathcal{F}_\tau = \mathcal{F}_{k_0}$.

(c) If τ, σ are two $\mathbb{F}$-stopping times, prove that $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$. Moreover, show that $\{\sigma \le \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$, and $\{\sigma = \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$.

(d) Let Y be an integrable random variable. Prove that

$$E[Y \mid \mathcal{F}_\tau] \mathbb{1}_{\{\tau = k\}} = E[Y \mid \mathcal{F}_k] \mathbb{1}_{\{\tau = k\}} \ P\text{-a.s. for all } k \in \{0, 1, \dots, T\},$$

or, equivalently,

$$E[Y \mid \mathcal{F}_\tau] = \sum_{k=0}^{T} \mathbb{1}_{\{\tau = k\}} E[Y \mid \mathcal{F}_k] \ \ P\text{-a.s.}$$

Solution 4.1

(a) We check the requirements for σ-algebra:

- $\Omega \in \mathcal{F}_\tau$ because $\Omega \cap \{\tau \le k\} = \{\tau \le k\} \in \mathcal{F}_k$ for all $k \in \{0, 1, \dots, T\}$.
- If $A \in \mathcal{F}_\tau$, then, for all $k \in \{0, 1, \dots, T\}$, it holds that

  $$A^c \cap \{\tau \le k\} = \{\tau \le k\} \cap (A^c \cup \{\tau \le k\}^c) = \{\tau \le k\} \cap (A \cap \{\tau \le k\})^c \in \mathcal{F}_k,$$

  so that $A^c \in \mathcal{F}_\tau$.
- If $A_n \in \mathcal{F}_\tau$, $n \in \mathbb{N}$, we have

  $$\left(\bigcup_{n=1}^{\infty} A_n \right) \cap \{\tau \le k\} = \bigcup_{n=1}^{\infty} (A_n \cap \{\tau \le k\}) \in \mathcal{F}_k$$

  for all $k \in \{0, 1, \dots, T\}$, and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_\tau$.

(b) Let $A \in \mathcal{F}_\sigma$ and $k \in \{0, 1, \ldots, T\}$, then we have

$$A \cap \{\tau \leq k\} = (\underbrace{A \cap \{\sigma \leq k\}}_{\in \mathcal{F}_k}) \cap \{\tau \leq k\} \in \mathcal{F}_k$$

because $A \in \mathcal{F}_\sigma$, and the assumption $\tau \leq \sigma$ implies that $\{\tau \leq k\} \subseteq \{\sigma \leq k\}$. This shows that $A \in \mathcal{F}_\tau$, and thus $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$ by arbitrariness of $A \in \mathcal{F}_\sigma$.

Now, if $\tau \equiv k_0$ for a fixed $k_0 \in \{0, 1, \ldots, T\}$, then $\mathcal{F}_\tau \subset \mathcal{F}_{k_0}$ and $\mathcal{F}_{k_0} \subset \mathcal{F}_\tau$, which yields to the desired equality $\mathcal{F}_{k_0} = \mathcal{F}_\tau$.

(c) Part (b) gives that $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ since $\sigma \wedge \tau \leq \sigma$ and $\sigma \wedge \tau \leq \tau$. Suppose next that $A \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. We observe that

$$\begin{aligned} A \cap \{\sigma \wedge \tau \leq k\} &= A \cap \Big(\{\sigma \leq k\} \cup \{\tau \leq k\}\Big) \\ &= \Big(\underbrace{A \cap \{\sigma \leq k\}}_{\in \mathcal{F}_k}\Big) \cup \Big(\underbrace{A \cap \{\tau \leq k\}}_{\in \mathcal{F}_k}\Big) \in \mathcal{F}_k \end{aligned}$$

for all $k \in \{0, 1, \ldots, T\}$. This shows $\mathcal{F}_\sigma \cap \mathcal{F}_\tau \subseteq \mathcal{F}_{\sigma \wedge \tau}$ and hence $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

To prove the remaining claims, note that, for each $k \in \{0, 1, \ldots, T\}$,

$$\{\sigma \leq \tau\} \cap \{\tau \leq k\} = \bigcup_{i=0}^{k} \Big(\{\sigma \leq \tau\} \cap \{\tau = i\}\Big) = \bigcup_{i=0}^{k} \Big(\{\sigma \leq i\} \cap \{\tau = i\}\Big) \in \mathcal{F}_k.$$

Thus $\{\sigma \leq \tau\} \in \mathcal{F}_\tau$. Similarly, we have

$$\{\sigma \leq \tau\} \cap \{\sigma \leq k\} = \{\sigma \wedge k \leq \tau \wedge k\} \cap \{\sigma \leq k\} \in \mathcal{F}_k$$

because $\sigma \wedge k$ and $\tau \wedge k$ are both $\mathbb{F}$-stopping times, and so $\{\sigma \wedge k \leq \tau \wedge k\} \in \mathcal{F}_{\tau \wedge k} \subseteq \mathcal{F}_k$ by the previous step. Hence,

$$\{\sigma \leq \tau\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \wedge \tau}.$$

The last assertion follows from the fact that $\{\sigma = \tau\} = \{\sigma \leq \tau\} \cap \{\tau \leq \sigma\}$.

(d) Let us fix $k \in \{0, 1, \ldots, T\}$, and denote by LHS and RHS the left- and right-hand sides of

$$E\,[Y \,|\, \mathcal{F}_\tau]\, \mathbb{1}_{\{\tau = k\}} = E\,[Y \,|\, \mathcal{F}_k]\, \mathbb{1}_{\{\tau = k\}},$$

respectively. Now, note that RHS is $\mathcal{F}_k$-measurable since $\{\tau = k\} \in \mathcal{F}_k$. Moreover, $\{\tau = k\} \in \mathcal{F}_\tau$ by part (c), and thus the random variable $\mathbb{1}_{\{\tau = k\}}$ is $\mathcal{F}_\tau$-measurable. It follows that

$$\text{LHS} = E\left[Y \mathbb{1}_{\{\tau = k\}} \,\middle|\, \mathcal{F}_\tau\right] \; P\text{-a.s.}.$$

For any $A \in \mathcal{F}_\tau$ it holds that $A \cap \{\tau = k\} = (A \cap \{\tau \leq k\}) \cap \{\tau = k\} \in \mathcal{F}_k$, where $k \in \{0, 1, \dots, T\}$. Then,

$$\begin{aligned} E\left[Y \mathbb{1}_{\{\tau=k\}} \mathbb{1}_A\right] &= E\left[Y \mathbb{1}_{A \cap \{\tau=k\}}\right] = E\left[E\left[Y \,|\, \mathcal{F}_k\right] \mathbb{1}_{A \cap \{\tau=k\}}\right] \\ &= E\left[E\left[Y \,|\, \mathcal{F}_k\right] \mathbb{1}_{\{\tau=k\}} \mathbb{1}_A\right] = E\left[\mathrm{RHS} \mathbb{1}_A\right], \end{aligned}$$

which shows that

$$\mathrm{RHS} = E\left[Y \mathbb{1}_{\{\tau=k\}} \,\middle|\, \mathcal{F}_\tau\right] = \mathrm{LHS} \ P\text{-a.s.}$$

Exercise 4.2 Let (S_0, S_1) be the (discounted) trinomial model with $T = 1$. This is a special case of the multinomial model where $S_0^1 = s_0^1$, for $s_0^1 > 0$, $S_1^1 = Y S_0^1/(1+r)$, for some $r > -1$ and

$$Y_k = \begin{cases} 1+d & \text{with probability } p_1, \\ 1+m & \text{with probability } p_2, \\ 1+u & \text{with probability } p_3 \end{cases}$$

where $-1 < d < m < u$, and p_1, p_2, $p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$. The filtration $\mathbb{F}$ we consider is given by $\mathcal{F}_0 := \{\varnothing, \Omega\}$, $\mathcal{F}_1 := \sigma(Y)$.

(a) Assume that $d = -0.5$, $m = 0$, $u = 0.25$ and $r = 0$, and consider an arbitrary self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \theta)$. Show that if the total gain $G_1(\theta)$ at time $T = 1$ is non-negative P-a.s., then

$$P[G_1(\theta) = 0] = 1.$$

What does this property imply?

(b) Show that S^1 is arbitrage-free by constructing an *equivalent martingale measure* (EMM) for S^1.
Hint: A probability measure Q equivalent to P on $\mathcal{F}_1$ can be uniquely described by a probability vector $(q_1, q_2, q_3) \in (0,1)^3$ whose coordinates sum up to 1, where $q_k = Q\left[Y_1 = 1 + y_k\right]$, $k = 1$, 2, 3, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.

Solution 4.2

(a) Let us compute the total gain $G_1(\theta)$ at time $T = 1$:

$$\begin{aligned} G_1(\theta) = \theta_1^1 \Delta S_1^1 = \theta_1^1 (S_1^1 - S_0^1) &= \theta_1^1 S_0^1 \left(\frac{Y_1}{1+r} - 1 \right) \\ &= \theta_1^1 S_0^1 \times \begin{cases} \frac{d-r}{1+r} & \text{with probability } p_1, \\ \frac{m-r}{1+r} & \text{with probability } p_2, \\ \frac{u-r}{1+r} & \text{with probability } p_3.. \end{cases} \end{aligned}$$

Recall that $u - r = 0.25 > 0$ and $d - r = -0.5 < 0$. Hence $P[G_1(\theta) \geq 0] = 1$ if and only if $\theta_1^1 S_0^1 = 0$. As a result, we can conclude that

$$P[G_1(\theta) \geq 0] = 1 \quad \Longleftrightarrow \quad \theta_1^1 = 0 \quad \Longleftrightarrow \quad P[G_1(\theta) = 0] = 1.$$

Assume now that $V_0 = 0$ and note that in this case $V_1(\varphi) = G_1(\theta)$. The above argument proves that if $V_1(\varphi) \geq 0$ P-a.s., then $V_1(\varphi) = 0$ P-a.s., and by Proposition 1.1 in the lecture notes, we know that this is equivalent to saying that S^1 is arbitrage-free.

(b) Let $(q_1, q_2, q_3) \in (0,1)^3$ be a probability vector and Q be defined by

$$Q\left[Y_1 = 1 + y_k\right] := q_k, \; k = 1, 2, 3,$$

where $y_1 := d$, $y_2 := m$ and $y_3 := u$. Then S^1 is a Q-martingale if and only if S^1 is adapted to the considered filtration (note that the filtration generated Y is equivalently generated by S^1), integrable (the probability space is finite here, so all random variables are integrable), and

$$\begin{aligned}
E_Q\left[S_1^1\right] = S_0^1 &\iff E_Q\left[S_0^1 Y_1/(1+r)\right] = S_0^1 \\
&\iff E_Q\left[Y_1\right] = 1 + r \\
&\iff q_1 \times (1+d) + q_2 \times (1+m) + q_3 \times (1+u) = 1 + r \\
&\iff q_1 \times d + q_2 \times m + q_3 \times u = r \\
&\iff -0.5 q_1 + 0 q_2 + 0.25 q_3 = 0 \\
&\iff q_3 = 2 q_1.
\end{aligned}$$

Recall that in order to make Q a probability measure, we need to have $q_1 + q_2 + q_3 = 1$; hence choosing $q_1 = 0.25$, we obtain that $q_3 = 0.5$ and $q_2 = 0.25$. Noting that $q_1, q_2, q_3 \in (0,1)$, we can also observe that Q is a probability measure equivalent to P and thus an EMM for S^1.

More generally, we can set $q_1 := \alpha$ to get $q_3 = 2\alpha$ and $q_2 = 1 - q_1 - q_3 = 1 - 3\alpha$. Then q_1, q_2 and q_3 are all in $(0,1)$ if and only if $\alpha \in (0, \frac{1}{3})$.

Exercise 4.3 Let (S^0, S^1) be the (discounted) binomial model with $T = 1$, $p \in (0,1)$, and $u > 0 > d > -1$. Fix some $K > 0$, and define the functions $h_C, h_P : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}
h_C(x) &:= (x - K)^+ := \max\{0, x - K\}, \\
h_P(x) &:= (K - x)^+ := \max\{0, K - x\}.
\end{aligned}$$

The European options with payoff functions h_C and h_P are called the *European call option* and the *European put option*, respectively.

(a) Construct a self-financing strategy $\varphi^C \mathrel{\widehat{=}} (V_0^C, \vartheta^C)$ such that

$$V_1(\varphi^C) = h_C(S_1^1).$$

Write down explicitly the values of V_0^C and ϑ_1^C.

(b) Construct a self-financing strategy $\varphi^P \mathrel{\widehat{=}} (V_0^P, \vartheta^P)$ such that

$$V_1(\varphi^P) = h_P(S_1^1).$$

Write down explicitly the values of V_0^P and ϑ_1^P.

(c) Prove the *put–call parity* relation

$$V_0^P - V_0^C = K - S_0^1.$$

Solution 4.3

(a) Consider a self-financing strategy $\varphi^C \mathrel{\widehat{=}} (V_0^C, \vartheta^C)$. By definition,

$$V_1(\varphi^C) = V_0^C + \vartheta_1^C \Delta S_1^1.$$

Since (S^0, S^1) is the binomial model, we have that either $S_1^1 = (1+u)S_0^1$ or $S_1^1 = (1+d)S_0^1$. Also, since ϑ_1^C is $\mathcal{F}_0$-measurable, it is a constant (i.e. non-random). Thus, φ satisfies $V_1(\varphi^C) = h_C(S_1^1)$ if and only if

$$\begin{aligned} V_0^C + \vartheta_1^C u S_0^1 &= h_C\Big((1+u)S_0^1\Big), \\ V_0^C + \vartheta_1^C d S_0^1 &= h_C\Big((1+d)S_0^1\Big). \end{aligned}$$

Subtracting the two equalities and rearranging gives

$$\vartheta_1^C = \frac{h_C((1+u)S_0^1) - h_C((1+d)S_0^1)}{(u-d)S_0^1}.$$

It remains to find V_0^C, which we can do by substituting the value of ϑ_1^C into either of the two previous equalities (we choose the first one) to get

$$\begin{aligned} V_0^C &= h_C\Big((1+u)S_0^1\Big) - \vartheta_1^C u S_0^1 \\ &= h_C\Big((1+u)S_0^1\Big) - \frac{h_C((1+u)S_0^1) - h_C((1+d)S_0^1)}{(u-d)S_0^1} u S_0^1 \\ &= \frac{u}{u-d} h_C\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_C\Big((1+u)S_0^1\Big). \end{aligned}$$

Note. Since $\frac{u}{u-d} + \frac{-d}{u-d} = 1$ *and* $\frac{u}{u-d} \in (0,1)$, *we can also write* $V_0^C = E^*[h_C(S_1^1)]$, *where* E^* *denotes the expectation under the "risk-neutral" probability measure* P^* *given by*

$$P^*[S_1^1 = (1+d)S_0^1] = \frac{u}{u-d}, \qquad P^*[S_1^1 = (1+u)S_0^1] = 1 - \frac{u}{u-d} = \frac{-d}{u-d}.$$

(b) The same reasoning as in part (a) yields

$$\vartheta_1^P = \frac{h_P((1+u)S_0^1) - h_P((1+d)S_0^1)}{(u-d)S_0^1},$$
$$V_0^P = \frac{u}{u-d} h_P\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_P\Big((1+u)S_0^1\Big).$$

Note. For the same risk-neutral probability measure P^* as in part (a), we can write

$$V_0^P = E^*[h_P(S_1^1)].$$

(c) First we compute, for $x \in \mathbb{R}$,

$$h_P(x) - h_C(x) = \max\{0, K - x\} - \max\{0, x - K\} = K - x.$$

Using this together with parts (a) and (b) yields

$$\begin{aligned}
V_0^P - V_0^C &= \frac{u}{u-d} h_P\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_P\Big((1+u)S_0^1\Big) \\
&\quad - \frac{u}{u-d} h_C\Big((1+d)S_0^1\Big) - \frac{-d}{u-d} h_C\Big((1+u)S_0^1\Big) \\
&= \frac{u}{u-d}\Big(K - (1+d)S_0^1\Big) + \frac{-d}{u-d}\Big(K - (1+u)S_0^1\Big) \\
&= K - S_0^1,
\end{aligned}$$

as required.

Alternatively, we could use the expectation under the risk-neutral measure to get

$$V_0^P - V_0^C = E^*[h_P(S_1^1) - h_C(S_1^1)] = E^*[K - S_1^1] = K - E^*[S_1^1].$$

We then compute

$$\begin{aligned}
E^*[S_1^1] &= (1+d)S_0^1 P^*[S_1^1 = (1+d)S_0^1] + (1+u)S_0^1 P^*[S_1^1 = (1+u)S_0^1] \\
&= (1+d)S_0^1 \frac{u}{u-d} + (1+u)S_0^1 \frac{-d}{u-d} \\
&= S_0^1,
\end{aligned}$$

and hence

$$V_0^P - V_0^C = K - S_0^1,$$

as required.