Mathematical Foundations for Finance

Exercise Sheet 4

Please hand in your solutions by 12:00 on Wednesday, October 25 via the course homepage.

Exercise 4.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. For any stopping time τ, we define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \colon A \cap \{\tau \le k\} \in \mathcal{F}_k \text{ for all } k = 0, 1, \dots, T\}.$$

(a) Show that $\mathcal{F}_\tau$ is a σ-algebra.

(b) Suppose σ, τ are two stopping times with $\sigma(\omega) \le \tau(\omega)$ for all $\omega \in \Omega$. Show that $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. Conclude that if $\tau \equiv k_0$ for a fixed $k_0 \in \{0, 1, \dots, T\}$, then we have that $\mathcal{F}_\tau = \mathcal{F}_{k_0}$.

(c) If τ, σ are two stopping times, prove that $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$. Moreover, show that $\{\sigma \le \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$, and $\{\sigma = \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$.

(d) Let Y be an integrable random variable. Prove that

$$E[Y \mid \mathcal{F}_\tau] \mathbb{1}_{\{\tau = k\}} = E[Y \mid \mathcal{F}_k] \mathbb{1}_{\{\tau = k\}} \ P\text{-a.s. for all } k \in \{0, 1, \dots, T\},$$

or, equivalently,

$$E[Y \mid \mathcal{F}_\tau] = \sum_{k=0}^{T} \mathbb{1}_{\{\tau = k\}} E[Y \mid \mathcal{F}_k] \ \ P\text{-a.s.}$$

Exercise 4.2 Let (S^0, S^1) be the (discounted) trinomial model with $T = 1$. This is a special case of the multinomial model where $S_0^1 = s_0^1$, for $s_0^1 > 0$, $S_1^1 = Y S_0^1/(1+r)$, for some $r > -1$ and

$$Y = \begin{cases} 1+d & \text{with probability } p_1, \\ 1+m & \text{with probability } p_2, \\ 1+u & \text{with probability } p_3, \end{cases}$$

where $-1 < d < m < u$, and p_1, p_2, $p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$. The filtration $\mathbb{F}$ we consider is given by $\mathcal{F}_0 := \{\varnothing, \Omega\}$, $\mathcal{F}_1 := \sigma(Y)$.

(a) Assume that $d = -0.5$, $m = 0$, $u = 0.25$ and $r = 0$ and consider an arbitrary self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \theta)$. Show that if the total gain $G_1(\theta)$ at time

$T = 1$ is non-negative P-a.s., then

$$P[G_1(\theta) = 0] = 1.$$

What does this property imply?

(b) Show that S^1 is arbitrage-free by constructing an *equivalent martingale measure* (EMM) for S^1.
Hint: A probability measure Q equivalent to P on $\mathcal{F}_1$ can be uniquely described by a probability vector $(q_1, q_2, q_3) \in (0,1)^3$ whose coordinates sum up to 1, where $q_k := Q[Y_1 = 1 + y_k]$, $k = 1,\ 2,\ 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.

Exercise 4.3 Let (S^0, S^1) be the (discounted) binomial model with $T = 1$, $p \in (0,1)$, and $u > 0 > d > -1$. For some $K > 0$, we define the functions $h_C, h_P : \mathbb{R} \to \mathbb{R}$ by

$$h_C(x) := (x - K)^+ := \max\{0, x - K\},$$
$$h_P(x) := (K - x)^+ := \max\{0, K - x\}.$$

The European options with payoff functions h_C and h_P are called the *European call option* and the *European put option*, respectively.

(a) Construct a self-financing strategy $\varphi^C \widehat{=} (V_0^C, \vartheta^C)$ such that

$$V_1(\varphi^C) = h_C(S_1^1).$$

Write down explicitly the values of V_0^C and ϑ_1^C.

(b) Construct a self-financing strategy $\varphi^P \widehat{=} (V_0^P, \vartheta^P)$ such that

$$V_1(\varphi^P) = h_P(S_1^1).$$

Write down explicitly the values of V_0^P and ϑ_1^P.

(c) Prove the *put–call parity* relation

$$V_0^P - V_0^C = K - S_0^1.$$