Mathematical Foundations for Finance

Exercise Sheet 5

Exercise 5.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and Y a random variable normally distributed such that $Y \sim \mathcal{N}(0,1)$.

(a) Fix a constant $\beta \in (0, \frac{1}{2})$, and consider the random variable

$$Z := \exp\left(-(\tfrac{1}{2} - \beta)Y - \frac{(\frac{1}{2} - \beta)^2}{2}\right).$$

Define the map $Q : \mathcal{F} \to \mathbb{R}$ by $Q[A] := E[Z\mathbb{1}_A]$. Prove that Q is a probability measure on $(\Omega, \mathcal{F})$, and that it is equivalent to P.

(b) Set

$$S_0^1 := e^{\beta} \qquad \text{and} \qquad S_1^1 := e^{Y}.$$

Prove that Q is an equivalent martingale measure for $S^1 = (S_0^1, S_1^1)$, with respect to the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$ given by $\mathcal{F}_0 := \{\varnothing, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

Hint: The statement $Q[A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{F}$ is equivalent to the statement $E_Q[U] = E[ZU]$ for all nonnegative random variables U.

(c) Now consider the market (S^0, S^1), where $S^0 \equiv 1$ represents a bank account and S^1 is as in part (b). Fix some $K > 0$ and define the function $C : \mathbb{R} \to \mathbb{R}$ by

$$C(x) = (x - K)^+ := \max\{x - K, 0\}.$$

Compute $V_0^C := E_Q[C(S_1^1)]$ in terms of the cumulative distribution function of a standard normal random variable.

Solution 5.1

(a) In order for Q to be a probability measure, we need to verify that

- $Q[A] \in [0,1]$ for all $A \in \mathcal{F}$;
- $Q[\varnothing] = 0$;
- $Q[\cup_{n=1}^{\infty} A_n] = \sum_{n=1}^{\infty} Q[A_n]$ for any disjoint family of sets $(A_n)_{n\in\mathbb{N}}$.

First, note that since Y is a standard normal random variable, Z is integrable (and nonnegative), and thus Q is a well-defined function with values in $[0, \infty)$.

Moreover,

$$Q[\Omega] = E[Z\mathbb{1}_\Omega] = E[Z] = \exp\left(-\frac{(\frac{1}{2}-\beta)^2}{2}\right) E\left[\exp\left(-(\tfrac{1}{2}-\beta)Y\right)\right] = 1,$$

since the moment-generating function of a standard normal random variable is $\phi(t) = E[e^{tY}] = \exp(\frac{t^2}{2})$. Also, we have $Q[\varnothing] = E[Z\mathbb{1}_\varnothing] = E[0] = 0$. Next, if $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then $\mathbb{1}_{\bigcup_{n=1}^\infty A_n} = \sum_{n=1}^\infty \mathbb{1}_{A_n}$, and so

$$Q\left[\bigcup_{n=1}^\infty A_n\right] = E\left[Z\mathbb{1}_{\bigcup_{n=1}^\infty A_n}\right] = E\left[\sum_{n=1}^\infty Z\mathbb{1}_{A_n}\right].$$

Since Z is nonnegative, then $0 \leqslant \sum_{n=1}^N Z\mathbb{1}_{A_n} \uparrow \sum_{n=1}^\infty Z\mathbb{1}_{A_n}$ as $N \to \infty$, and thus the monotone convergence theorem implies that

$$E\left[\sum_{n=1}^\infty Z\mathbb{1}_{A_n}\right] = \lim_{N\to\infty} E\left[\sum_{n=1}^N Z\mathbb{1}_{A_n}\right] = \lim_{N\to\infty}\sum_{n=1}^N E[Z\mathbb{1}_{A_n}] = \sum_{n=1}^\infty E[Z\mathbb{1}_{A_n}].$$

Hence, we have

$$Q\left[\bigcup_{n=1}^\infty A_n\right] = \sum_{n=1}^\infty Q[A_n].$$

We can conclude that Q is a probability measure on $(\Omega, \mathcal{F})$.

It remains to show that $Q \approx P$. To this end, let $A \in \mathcal{F}$ with $P[A] = 0$. Then $Z\mathbb{1}_A = 0$ P-a.s, and thus

$$Q[A] = E[Z\mathbb{1}_A] = E[0] = 0.$$

Hence, $Q \ll P$. Conversely, suppose that $A \in \mathcal{F}$ with $Q[A] = 0$. This means that $E[Z\mathbb{1}_A] = 0$. Since $Z\mathbb{1}_A$ is nonnegative, then $Z\mathbb{1}_A = 0$ P-a.s. Also, since $Z > 0$, then $Z\mathbb{1}_A = 0$ exactly when $\mathbb{1}_A = 0$, and so $\mathbb{1}_A = 0$ P-a.s, i.e. $P[A] = 0$. It follows that $P \ll Q$, and hence $Q \approx P$, as required.

(b) Since $Q \approx P$ by part (a), it remains to show that S^1 is a Q-martingale. It is immediate that S^1 is $\mathbb{F}$-adapted. Also, since $S_1^1 \geqslant 0$, we have $E[|S_1^1|] = E[S_1^1]$, and by the hint,

$$\begin{aligned} E_Q[S_1^1] &= E[ZS_1^1] = E\left[\exp\left((\tfrac{1}{2}+\beta)Y - \frac{(\frac{1}{2}-\beta)^2}{2}\right)\right] \\ &= \exp\left(-\frac{(\frac{1}{2}-\beta)^2}{2}\right) E\left[\exp\left((\tfrac{1}{2}+\beta)Y\right)\right] = \exp\left(\frac{(\frac{1}{2}+\beta)^2 - (\frac{1}{2}-\beta)^2}{2}\right) \\ &= e^\beta < \infty, \end{aligned}$$

using again that $E[e^{tY}] = \exp(\frac{t^2}{2})$. Thus, S^1 is Q-integrable. Finally, we note that since $\mathcal{F}_0$ is the trivial σ-field, then by Exercise 1.3(c),

$$E_Q[S_1^1 \mid \mathcal{F}_0] = E_Q[S_1^1] = e^\beta = S_0^1.$$

This completes the proof.

(c) We have that

$$\begin{aligned} V_0^C &= E_Q[C(S_1^1)] = E_Q[(S_1^1 - K)^+] = E[Z(S_1^1 - K)^+] \\ &= \exp\left(-\frac{(\frac{1}{2}-\beta)^2}{2}\right) E\left[\exp\left(-(\tfrac{1}{2}-\beta)Y\right)(e^Y - K)\mathbb{1}_{\{e^Y > K\}}\right]. \end{aligned}$$

Since the distribution of Y has density $f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})$, and also since $\{e^Y > K\} = \{Y > \log K\}$, we can write

$$E\left[\exp\left(-(\tfrac{1}{2}-\beta)Y\right)(e^Y - K)\mathbb{1}_{\{e^Y > K\}}\right] = \frac{1}{\sqrt{2\pi}}\int_{\log K}^{\infty} e^{-(\frac{1}{2}-\beta)y}(e^y - K)e^{-\frac{y^2}{2}}\,\mathrm{d}y.$$

The integrand can be rewritten as

$$\begin{aligned} e^{-(\frac{1}{2}-\beta)y}(e^y - K)e^{-\frac{y^2}{2}} &= e^{(\frac{1}{2}+\beta)y - \frac{y^2}{2}} - Ke^{-(\frac{1}{2}-\beta)y - \frac{y^2}{2}} \\ &= e^{-\frac{(y-(\beta+\frac{1}{2}))^2}{2}} e^{\frac{(\beta+\frac{1}{2})^2}{2}} - Ke^{-\frac{(y-(\beta-\frac{1}{2}))^2}{2}} e^{\frac{(\beta-\frac{1}{2})^2}{2}}, \end{aligned}$$

and so

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{\log K}^{\infty} e^{-(\frac{1}{2}-\beta)y}(e^y - K)e^{-\frac{y^2}{2}}\,\mathrm{d}y &= e^{\frac{(\beta+\frac{1}{2})^2}{2}}\int_{\log K}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-(\beta+\frac{1}{2}))^2}{2}}\,\mathrm{d}y \\ &\quad - Ke^{\frac{(\beta-\frac{1}{2})^2}{2}}\int_{\log K}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-(\beta-\frac{1}{2}))^2}{2}}\,\mathrm{d}y \\ &= e^{\frac{(\beta+\frac{1}{2})^2}{2}} P[Y + \beta + \tfrac{1}{2} > \log K] \\ &\quad - Ke^{\frac{(\beta-\frac{1}{2})^2}{2}} P[Y + \beta - \tfrac{1}{2} > \log K], \end{aligned}$$

since the above two integrands are the densities of the random variables $Y+\beta+\frac{1}{2}$ and $Y + \beta - \frac{1}{2}$, respectively. Letting $\Phi(x) := P[Y \leqslant x]$ denote the cumulative distribution function of a standard normal random variable, we can rewrite the above as

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{\log K}^{\infty} e^{-(\frac{1}{2}-\beta)y}(e^y - K)e^{-\frac{y^2}{2}}\,\mathrm{d}y &= e^{\frac{(\beta+\frac{1}{2})^2}{2}}\Phi(-\log K + \beta + \tfrac{1}{2}) \\ &\quad - e^{\frac{(\beta-\frac{1}{2})^2}{2}} K\Phi(-\log K + \beta - \tfrac{1}{2}). \end{aligned}$$

Remembering that V_0^C is the product of $\exp(-\frac{(\frac{1}{2}-r)^2}{2})$ and the above difference, we get

$$\begin{aligned} V_0^C &= e^{\frac{-(\frac{1}{2}-\beta)^2 + (\beta+\frac{1}{2})^2}{2}}\Phi(-\log K + \beta + \tfrac{1}{2}) - K\Phi(-\log K + \beta - \tfrac{1}{2}) \\ &= e^{\beta}\Phi(-\log K + \beta + \tfrac{1}{2}) - K\Phi(-\log K + \beta - \tfrac{1}{2}). \end{aligned}$$

This completes the problem.

Exercise 5.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, and $T \in \mathbb{N}$. Let $Y = (Y_k)_{k=0,1,\dots,T}$, be an integrable, $\mathbb{F}$-adapted process, and define the $\mathbb{F}$-adapted process $U = (U_t)_{k=0,1,\dots,T}$ by

$$\begin{aligned} U_T &= Y_T \\ U_k &= \max\Big(Y_k, E\left[U_{k+1} \,|\, \mathcal{F}_k\right]\Big) \quad \text{for } k = 0, 1, \dots, T-1. \end{aligned}$$

The process U is called the Snell envelope of Y. For simplicity, we suppose that $\mathcal{F}_0$ in $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ is the trivial σ-algebra $\{\emptyset, \Omega\}$.

(a) Show that the Snell envelope U of Y is the smallest supermartingale dominating Y, in the sense that if $V = (V_k)_{k=0,1,\dots,T}$ is a supermartingale with $V_k \geq Y_k$ P-a.s. for all $k = 0, 1, \dots, T$, then we have $V_k \geq U_k$ P-a.s. for all $k = 0, 1, \dots, T$ as well.
Hint: Proceed by backward induction.

(b) Show that if Y is a supermartingale, then $U_k = Y_k$ P-a.s. for all $k = 0, 1, \dots, T$, and if Y is a submartingale, then $U_k = E\left[Y_T \,|\, \mathcal{F}_k\right]$ P-a.s. for all $k = 0, 1, \dots, T$.
Hint: Proceed by backward induction.

(c) Using your result from (b), show that if Y is a submartingale, then U is a martingale.

(d) Let τ be an $\mathbb{F}$-stopping time. Show that the stopped process $U^\tau = (U_{k\wedge\tau})_{k=0,1,\dots,T}$ is a supermartingale.

Let us now define

$$\tau^* := \inf\Big\{k \in \{0, 1, \dots, T\} \,:\, U_k = Y_k\Big\}.$$

(e) Show that τ^* is an $\mathbb{F}$-stopping time. Furthermore, show that the stopped process U^{τ^*} is a martingale and, in particular, that $U_0 = E\left[Y_{\tau^*}\right]$.

Solution 5.2

(a) Adaptedness of U is immediate since the maximum of a two measurable functions is again a measurable function. Integrability follows by backward induction:

- $Y_T \in L^1(P)$ by assumption.
- Suppose now that $U_n \in L^1(P)$ for some $n \in \{1, \dots, T\}$; we want to show that this implies $U_{n-1} \in L^1(P)$. We have

$$\begin{aligned} E\left[|U_{n-1}|\right] &= E\Big[\big|\max\Big(Y_{n-1}, E\left[U_n \,|\, \mathcal{F}_{n-1}\right]\Big)\big|\Big] \\ &\leq E\Big[\max\Big(|Y_{n-1}|, |E\left[U_n \,|\, \mathcal{F}_{n-1}\right]|\Big)\Big] \\ &\leq E\Big[\max\Big(|Y_{n-1}|, E\left[|U_n| \,|\, \mathcal{F}_{n-1}\right]\Big)\Big] \leq E\left[|Y_{n-1}|\right] + E\left[|U_n|\right] < \infty, \end{aligned}$$

where the second inequality uses Jensen's inequality for conditional expectations.

Moreover, by the definition of the process U, $U_T = Y_T$ and $U_k \geq E\left[U_{k+1} \,|\, \mathcal{F}_k\right]$ for $k = 0, 1, \ldots, T-1$, so that the Snell envelope U of the process Y is a supermartingale, and $U_k \geq Y_k$ for all $k = 0, 1, \ldots, T$ by construction. It remains to show that U is the smallest such process. Let $V = (V_k)_{k=0,1,\ldots,T}$ be any other supermartingale dominating Y. We have to show that V dominates U as well. We again proceed by backward induction:

- First, since $U_T = Y_T$ and V dominates Y, we have that $V_T \geq U_T$.
- Suppose now that $V_n \geq U_n$ for some $n \in \{1, \ldots, T\}$; we want to show that this implies $V_{n-1} \geq U_{n-1}$. We have

$$V_{n-1} \geq E\left[V_n \,|\, \mathcal{F}_{n-1}\right] \geq E\left[U_n \,|\, \mathcal{F}_{n-1}\right],$$

  where the first inequality uses the supermartingale property of V and the second one uses the induction hypothesis. Furthermore, since V dominates Y, we also have that $V_{n-1} \geq Y_{n-1}$. Combining the two results, we get

$$V_{n-1} \geq \max\Big(Y_{n-1}, E\left[U_n \,|\, \mathcal{F}_{n-1}\right]\Big) = U_{n-1}$$

  as desired.

(b) We proceed by backward induction in both cases. Let us first assume that Y is a supermartingale:

- By the definition of U, we have that $U_T = Y_T$.
- Suppose now that $U_n = Y_n$ for some $n \in \{1, \ldots, T\}$; we want to show that this implies $U_{n-1} = Y_{n-1}$. We have

$$U_{n-1} = \max\Big(Y_{n-1}, E\left[U_n \,|\, \mathcal{F}_{n-1}\right]\Big) = \max\Big(Y_{n-1}, E\left[Y_n \,|\, \mathcal{F}_{n-1}\right]\Big) = Y_{n-1}.$$

  The last equality follows since $E\left[Y_n \,|\, \mathcal{F}_{n-1}\right] \leq Y_{n-1}$ by the assumption that Y is a supermartingale.

Let Y be a submartingale:

- Since Y is $\mathbb{F}$-adapted, we have that $U_T = Y_T = E\left[Y_T \,|\, \mathcal{F}_T\right]$.
- Suppose now that $U_n = E\left[Y_T \,|\, \mathcal{F}_n\right]$ for some $n \in \{1, \ldots, T\}$; we want to show that this implies $U_{n-1} = E\left[Y_T \,|\, \mathcal{F}_{n-1}\right]$. We have

$$\begin{aligned} U_{n-1} &= \max\Big(Y_{n-1}, E\left[U_n \,|\, \mathcal{F}_{n-1}\right]\Big) = \max\Big(Y_{n-1}, E\left[E\left[Y_T \,|\, \mathcal{F}_n\right] \,|\, \mathcal{F}_{n-1}\right]\Big) \\ &= \max\Big(Y_{n-1}, E\left[Y_T \,|\, \mathcal{F}_{n-1}\right]\Big) = E\left[Y_T \,|\, \mathcal{F}_{n-1}\right]. \end{aligned}$$

  The third equality uses the tower property of conditional expectations and the last one that $E\left[Y_T \,|\, \mathcal{F}_{n-1}\right] \geq Y_{n-1}$ by the assumption that Y is a submartingale.

(c) Since we know that U is a supermartingale, it is integrable and $\mathbb{F}$-adapted. Furthermore, we know from (b) that $U_k = E[Y_T \mid \mathcal{F}_k]$ for all $k = 0, 1, \ldots, T$. Therefore, we have for all $k = 0, 1, \ldots, T-1$ that

$$E[U_{k+1} \mid \mathcal{F}_k] = E[E[Y_T \mid \mathcal{F}_{k+1}] \mid \mathcal{F}_k] = E[Y_T \mid \mathcal{F}_k] = U_k,$$

where the second equality uses the tower property of conditional expectations. This is the martingale property of U, so U is in fact a martingale.

(d) Note that we have for all $k \in \{0, 1, \ldots, T-1\}$ that

$$U_{(k+1)\wedge\tau} - U_{k\wedge\tau} = \mathbb{1}_{\{k+1\leq\tau\}}(U_{k+1} - U_k). \tag{1}$$

The supermartingale property of U^τ now immediately follows from the supermartingale property of U:

$$\begin{aligned} E\left[U_{(k+1)\wedge\tau} - U_{k\wedge\tau} \,\middle|\, \mathcal{F}_k\right] &= E\left[\mathbb{1}_{\{k+1\leq\tau\}}(U_{k+1} - U_k) \,\middle|\, \mathcal{F}_k\right] \\ &= \mathbb{1}_{\{k+1\leq\tau\}} E[U_{k+1} - U_k \mid \mathcal{F}_k] \leq 0, \end{aligned}$$

where the third equality uses that $\mathbb{1}_{\{k+1\leq\tau\}} = 1 - \mathbb{1}_{\{\tau\leq k\}}$ is $\mathcal{F}_k$-measurable and the last inequality holds by our assumption that U is a supermartingale. Adaptedness and integrability of a general stopped process follow from its stochastic integral representation analogous to the solution of Exercise 3.1 (c).

(e) First note that since $U_T = Y_T$ by construction, $\tau^* \leq T$. For $k = 0, 1, \ldots, T$, the set

$$\begin{aligned} \{\tau^* \leq k\}^c = \{\tau^* > k\} &= \{Y_0 < U_0, \ldots, Y_k < U_k\} \\ &= \{Y_0 - U_0 < 0, \ldots, Y_k - U_k < 0\} \end{aligned}$$

is in $\mathcal{F}_k$ since both U and Y are $\mathbb{F}$-adapted processes and the difference of two real-valued $\mathcal{F}_k$-measurable random variables is a real-valued $\mathcal{F}_k$-measurable random variable. Hence, $\{\tau^* \leq k\} \in \mathcal{F}_k$ for $k = 0, 1, \ldots, T$ and τ^* is an $\mathbb{F}$-stopping time.

Now we show that U^{τ^*} is a not only a supermartingale, but actually a true martingale. Note that $U_k \mathbb{1}_{\{k+1\leq\tau^*\}} = E[U_{k+1} \mid \mathcal{F}_k] \mathbb{1}_{\{k+1\leq\tau^*\}}$ by the definition of Snell envelope and the definition of the stopping time τ^*: If $k < \tau^* \iff k+1 \leq \tau^*$, then we must have $U_k \neq Y_k$ which implies $U_k > Y_k$ and thus giving $U_k = E[U_{k+1} \mid \mathcal{F}_k]$. Hence, using (1), we have

$$U_{(k+1)\wedge\tau^*} - U_{k\wedge\tau^*} = \mathbb{1}_{\{k+1\leq\tau^*\}}(U_{k+1} - U_k) = \mathbb{1}_{\{k+1\leq\tau^*\}}\Big(U_{k+1} - E[U_{k+1} \mid \mathcal{F}_k]\Big).$$

This then implies that

$$E\left[U_{(k+1)\wedge\tau^*} - U_{k\wedge\tau^*} \,\middle|\, \mathcal{F}_k\right] = \mathbb{1}_{\{k+1\leq\tau^*\}} E[U_{k+1} - E[U_{k+1} \mid \mathcal{F}_k] \mid \mathcal{F}_k] = 0,$$

which is the martingale property of U^{τ^*}. In particular, we have $E[U_{T\wedge\tau^*}] = U_0$ by Corollary I.3.2 in the lecture notes. Moreover, since $\tau^* \leq T$, we have

$$E[Y_{\tau^*}] = E[Y_{T\wedge\tau^*}] = E[U_{T\wedge\tau^*}] = E[U_{\tau^*}] = U_0.$$