# Mathematical Foundations for Finance

## Exercise Sheet 5

*Please hand in your solutions by 12:00 on Wednesday, November 1 via the course homepage.*

**Exercise 5.1** Let $(\Omega, \mathcal{F}, P)$ be a probability space and $Y$ a random variable normally distributed such that $Y \sim \mathcal{N}(0,1)$.

(a) Fix a constant $\beta \in (0, \frac{1}{2})$, and consider the random variable

$$Z := \exp\left(-(\tfrac{1}{2} - \beta)Y - \frac{(\frac{1}{2} - \beta)^2}{2}\right).$$

Define the map $Q : \mathcal{F} \to \mathbb{R}$ by $Q[A] := E[Z\mathbb{1}_A]$. Prove that $Q$ is a probability measure on $(\Omega, \mathcal{F})$, and that it is equivalent to $P$.

(b) Set

$$S_0^1 := e^{\beta} \qquad \text{and} \qquad S_1^1 := e^Y.$$

Prove that $Q$ is an equivalent martingale measure for $S^1 = (S_0^1, S_1^1)$, with respect to the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$ given by $\mathcal{F}_0 := \{\varnothing, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

*Hint: The statement $Q[A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{F}$ is equivalent to the statement $E_Q[U] = E[ZU]$ for all nonnegative random variables $U$.*

(c) Now consider the market $(S^0, S^1)$, where $S^0 \equiv 1$ represents a bank account and $S^1$ is as in part (b). Fix some $K > 0$ and define the function $C : \mathbb{R} \to \mathbb{R}$ by

$$C(x) = (x - K)^+ := \max\{x - K, 0\}.$$

Compute $V_0^C := E_Q[C(S_1^1)]$ in terms of the cumulative distribution function of a standard normal random variable.

**Exercise 5.2** Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, and $T \in \mathbb{N}$. Let $Y = (Y_k)_{k=0,1,\dots,T}$, be an integrable, $\mathbb{F}$-adapted process, and define the $\mathbb{F}$-adapted process $U = (U_t)_{k=0,1,\dots,T}$ by

$$\begin{aligned} U_T &= Y_T \\ U_k &= \max\Big(Y_k, E\left[U_{k+1}\,|\,\mathcal{F}_k\right]\Big) \quad \text{for } k = 0, 1, \dots, T-1. \end{aligned}$$

The process $U$ is called the Snell envelope of $Y$. For simplicity, we suppose that $\mathcal{F}_0$ in $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ is the trivial $\sigma$-algebra $\{\emptyset, \Omega\}$.

(a) Show that the Snell envelope $U$ of $Y$ is the smallest supermartingale dominating $Y$, in the sense that if $V = (V_k)_{k=0,1,\dots,T}$ is a supermartingale with $V_k \geq Y_k$ $P$-a.s. for all $k = 0, 1, \dots, T$, then we have $V_k \geq U_k$ $P$-a.s. for all $k = 0, 1, \dots, T$ as well.
*Hint: Proceed by backward induction.*

(b) Show that if $Y$ is a supermartingale, then $U_k = Y_k$ $P$-a.s. for all $k = 0, 1, \dots, T$, and if $Y$ is a submartingale, then $U_k = E[Y_T \,|\, \mathcal{F}_k]$ $P$-a.s. for all $k = 0, 1, \dots, T$.
*Hint: Proceed by backward induction.*

(c) Using your result from (b), show that if $Y$ is a submartingale, then $U$ is a martingale.

(d) Let $\tau$ be an $\mathbb{F}$-stopping time. Show that the stopped process $U^\tau = (U_{k\wedge\tau})_{k=0,1,\dots,T}$ is a supermartingale.

Let us now define

$$\tau^* := \inf\Big\{k \in \{0, 1, \dots, T\} \,:\, U_k = Y_k\Big\}.$$

(e) Show that $\tau^*$ is an $\mathbb{F}$-stopping time. Furthermore, show that the stopped process $U^{\tau^*}$ is a martingale and, in particular, that $U_0 = E[Y_{\tau^*}]$.