Mathematical Foundations for Finance

Exercise Sheet 6

Please hand in your solutions by 12:00 on Wednesday, November 8 via the course homepage.

Exercise 6.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. Let $X = (X_k)_{k\in\mathbb{N}_0}$ be an adapted and integrable process.

(a) Find the *Doob decomposition* of X. In other words, prove that there exist a martingale $M = (M_k)_{k\in\mathbb{N}_0}$ and an integrable and predictable process $A = (A_k)_{k\in\mathbb{N}_0}$ that are both null at zero, and such that

$$X = X_0 + M + A \ P\text{-a.s.}$$

Hint: You may define $M_k := \sum_{j=1}^{k}(X_j - E[X_j \mid \mathcal{F}_{j-1}])$, *for* $k \in \mathbb{N}$.

(b) Prove that M and A are unique up to P-a.s. equality.

Exercise 6.2 Let $W = (W_t)_{t\geq 0}$ and $W' = (W'_t)_{t\geq 0}$ be two *independent* Brownian motions (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$. Show that

(a) $W^1 := -W$ is a BM.

(b) $W^2_t := W_{T+t} - W_T$, for $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha W + \sqrt{1-\alpha^2}W'$ is a BM for any $\alpha \in [0, 1]$.

(d) Show that the independence of W and W' in (c) cannot be omitted, i.e., if W and W' are *not* independent, then W^3 need not be a BM. Give two examples.

Exercise 6.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t\geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g\colon \mathbb{R}_+ \to \mathbb{R}$, show that the process

$$\Big(M_t + g(t)\Big)_{t\geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.