Mathematical Foundations for Finance

Exercise Sheet 7

Please hand in your solutions by 12:00 on Wednesday, November 15 via the course homepage.

Exercise 7.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales, but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: You may use the results from Exercise 6.3(b) and Exercise 6.3(a), respectively.

(b) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.

Exercise 7.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. For any constants a, $b \in \mathbb{R}$ such that $a < 0 < b$, consider the function $\tau : \Omega \to [0, \infty]$ given by

$$\tau := \inf\{t \geqslant 0 : W_t \notin [a, b]\}.$$

(a) Show that τ is a stopping time.
Hint: You may use the right-continuity of the filtration $\mathbb{F}$.

(b) Prove that $E[W_\tau] = 0$.
Hint: You may apply the dominated convergence theorem.

(c) Compute $P[W_\tau = a]$.
Hint: You may use the result from (b).