Mathematical Foundations for Finance

Exercise Sheet 8

Exercise 8.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) For some constants $S_0 > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$, we define the *geometric Brownian motion* $S = (S_t)_{t\geqslant 0}$ as follows

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

Compute $\lim_{t\to\infty} S_t$ when $\mu \neq \frac{\sigma^2}{2}$. Determine whether the limit exists if $\mu = \frac{\sigma^2}{2}$.
Hint: You may use the law of the iterated logarithm.

(b) Prove that

$$E[W_t^3 - W_s^3 \mid \mathcal{F}_s] = 3(t-s)W_s \ P\text{-a.s., for } 0 \leq s < t.$$

Hint: You may compute $E[(W_t - W_s)^3 \mid \mathcal{F}_s]$.

Solution 8.1

(a) For any $t \geq 0$, we can rewrite S_t as

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{2t\log\log t}\frac{W_t}{\sqrt{2t\log\log t}}\right) \\ &= S_0 \exp\left(\sqrt{2t\log\log t}\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{t}{\sqrt{2t\log\log t}} + \sigma\frac{W_t}{\sqrt{2t\log\log t}}\right)\right). \end{aligned}$$

Since

$$\lim_{t\to\infty}\sqrt{2t\log\log t} = +\infty \quad \text{and} \quad \lim_{t\to\infty}\frac{t}{\sqrt{2t\log\log t}} = +\infty,$$

and by the law of the iterated logarithm, it follows that:

- when $\mu > \frac{\sigma^2}{2}$,

$$\lim_{t\to\infty} S_t = +\infty \ P\text{-a.s.};$$

- when $\mu < \frac{\sigma^2}{2}$,

$$\lim_{t\to\infty} S_t = 0 \ P\text{-a.s.}$$

If $\mu = \frac{\sigma^2}{2}$, the limit $\lim_{t\to\infty} S_t$ does not exist since

$$P\Big(\liminf_{t\to\infty} S_t = 0\Big) = 1 \quad \text{and} \quad P\Big(\limsup_{t\to\infty} S_t = +\infty\Big) = 1.$$

(b) To simplify notation, we omit "P-a.s." from all equalities below. Let us fix some $0 \leq s < t$. Since $W_t - W_s$ is independent of $\mathcal{F}_s$, so is $(W_t - W_s)^3$. Hence,

$$E[(W_t - W_s)^3 \mid \mathcal{F}_s] = E[(W_t - W_s)^3].$$

Since $W_t - W_s \sim N(0, t-s)$, then $E[(W_t - W_s)^3] = 0$, and thus

$$\begin{aligned} E[W_t^3 - W_s^3 \mid \mathcal{F}_s] &= E[W_t^3 - W_s^3 \mid \mathcal{F}_s] - E[(W_t - W_s)^3 \mid \mathcal{F}_s] \\ &= E[3W_t^2 W_s - 3W_t W_s^2 \mid \mathcal{F}_s]. \end{aligned}$$

From the fact that W_t and W_s are normal random variables, we deduce that W_t, W_t^2, W_s, W_s^2, and all products, are integrable. Hence, we get

$$\begin{aligned} E[W_t^3 - W_s^3 \mid \mathcal{F}_s] &= 3W_s E[W_t^2 \mid \mathcal{F}_s] - 3W_s^2 E[W_t \mid \mathcal{F}_s] \\ &= 3W_s(W_s^2 + t - s) - 3W_s^3 \\ &= 3(t-s)W_s, \end{aligned}$$

where in the second step we have used that $(W)_{t\geqslant 0}$ and $(W_t^2 - t)_{t\geqslant 0}$ are martingales.

Exercise 8.2 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. On this space, let M a local martingale null at 0 that satisfies $\sup_{0\leq t\leq T}|M_t| \in L^2$ for some $T \in \mathbb{R}$.

(a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: You may use dominated convergence theorem.

(b) Let $[M]$ be the square bracket process of M. Prove that

$$E\Big[[M]_t - [M]_s \,\Big|\, \mathcal{F}_s\Big] = \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s] \ P\text{-a.s., for } 0 \leq s \leq t \leq T.$$

Hint: You may use that $\mathrm{Var}[X \,|\, \mathcal{G}] = E\Big[(X - E[X \,|\, \mathcal{G}])^2 \,\Big|\, \mathcal{G}\Big]$.

Solution 8.2

(a) The process M is adapted by definition since it is a local martingale. Moreover, for any $s \in [0, T]$, it holds that

$$|M_s|^2 \leq |M_T^*|^2, \text{ where } M_T^* := \sup_{0\leq u\leq T} |M_u|.$$

By assumption, $M_T^* \in L^2$ and thus M is square-integrable on $[0, T]$.

Now, let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for M. For every fixed $n \in \mathbb{N}$, we have that

$$E\left[M_{\tau_n\wedge t}\,|\,\mathcal{F}_s\right] = M_{\tau_n\wedge s}\ P\text{-a.s., for } 0 \le s \le t \le T, \tag{1}$$

because M is a local martingale. Since $|M_{\tau_n\wedge t}|$ is bounded from above by the integrable random variable M_T^*, for all $0 \le t \le T$, the dominated convergence theorem gives us that

$$\lim_{n\to\infty} E\left[M_{\tau_n\wedge t}\,|\,\mathcal{F}_s\right] = E\left[\lim_{n\to\infty} M_{\tau_n\wedge t}\,\middle|\,\mathcal{F}_s\right] = E\left[M_t\,|\,\mathcal{F}_s\right]\ P\text{-a.s.} \tag{2}$$

On the other hand, we have for the right-hand side of (1) that

$$\lim_{n\to\infty} M_{\tau_n\wedge s} = M_s\ P\text{-a.s.},$$

which, together with (2), gives us the martingale property for M on $[0, T]$ and concludes the proof.

(b) Since M is a square-integrable martingale on $[0, T]$, the square bracket process $[M]$ is integrable and $M^2 - [M]$ is a martingale according to Theorem V.1.1 in the lecture notes. Therefore, for all $0 \le s \le t \le T$, it holds that

$$\begin{aligned}
E\Big[[M]_t - [M]_s\,\Big|\,\mathcal{F}_s\Big] &= E\left[M_t^2 - M_s^2\,\middle|\,\mathcal{F}_s\right]\\
&= E\left[(M_t - M_s)^2\,\middle|\,\mathcal{F}_s\right]\\
&= E\left[\Big(M_t - E\left[M_t\,|\,\mathcal{F}_s\right]\Big)^2\,\middle|\,\mathcal{F}_s\right]\\
&= E\left[\Big(M_t - M_s + M_s - E\left[M_t\,|\,\mathcal{F}_s\right]\Big)^2\,\middle|\,\mathcal{F}_s\right]\\
&= E\left[\Big((M_t - M_s) - E\left[M_t - M_s\,|\,\mathcal{F}_s\right]\Big)^2\,\middle|\,\mathcal{F}_s\right]\\
&= \mathrm{Var}[M_t - M_s\,|\,\mathcal{F}_s]\ P\text{-a.s.}
\end{aligned}$$

Exercise 8.3 Let $(\Omega, \mathcal{F}, P)$ a probability space. We consider a sequence $(Y_k)_{k\in\mathbb{N}}$ of square-integrable and independent random variables and the filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \ldots, Y_k)$ for all $k \in \mathbb{N}$. We assume that $(Y_k)_{k\in\mathbb{N}}$ are identically distributed, with $\mu := E\left[Y_k\right] \in \mathbb{R}$ and $\sigma^2 := \mathrm{Var}[Y_k] > 0$, for $k \in \mathbb{N}$. Define the process $X = (X_n)_{n\in\mathbb{N}_0}$ by

$$X_n = \sum_{k=1}^{n} Y_k,\ \text{for } n \in \mathbb{N}_0.$$

Note that X is adapted to $\mathbb{F}$ and integrable.

(a) Derive the Doob decomposition of X. In other words, find the martingale $M = (M_n)_{n\in\mathbb{N}_0}$ and the predictable and integrable process $A = (A_n)_{n\in\mathbb{N}_0}$ that are both null at zero and such that $X = M + A$ P-a.s. Deduce that M and A are square-integrable.
Hint: We have done it in Exercise 6.1(a).

(b) Find the optional quadratic variation $[M] = ([M]_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.
Hint: You may use Theorem V.1.1 in the lecture notes, and in particular the condition $\Delta[M] = (\Delta M)^2$.

(c) Explicitly derive the predictable quadratic variation $\langle M\rangle = (\langle M\rangle_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.

Solution 8.3 To simplify notation, we omit "P-a.s." from all equalities below.

(a) Let us fix $n \in \mathbb{N}$. From Exercise 6.1(a), we know that

$$\begin{aligned} M_n &= \sum_{j=1}^{n} \Big(X_j - E\left[X_j \,|\, \mathcal{F}_{j-1}\right] \Big) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{j} \Big(Y_k - E\left[Y_k \,|\, \mathcal{F}_{j-1}\right] \Big) \\ &= \sum_{j=1}^{n} \Big(Y_j - E\left[Y_j \,|\, \mathcal{F}_{j-1}\right] \Big) \end{aligned}$$

since Y_k is $\mathcal{F}_{j-1}$-measurable for all $k \le j-1$. Moreover, Y_j is independent of $\mathcal{F}_{j-1}$, and thus

$$M_n = \sum_{j=1}^{n} \Big(Y_j - E[Y_j] \Big) = X_n - n\mu.$$

Hence,

$$A_n = X_n - M_n = n\mu.$$

We conclude that both M and A are square-integrable since so is the process X by assumption.

(b) Since the process M is a square-integrable martingale, Theorem V.1.1 from the lecture notes states that there exists a unique adapted increasing RCLL process $[M] = ([M]_n)_{n\in\mathbb{N}_0}$ null at 0 with $\Delta[M] = (\Delta M)^2$ and having the property that $M^2 - [M]$ is a local martingale. Hence, for each $n \in \mathbb{N}$, we have

$$\Delta[M]_n = (\Delta M_n)^2 = (M_n - M_{n-1})^2 = (Y_n - \mu)^2,$$

so that

$$[M]_n = \sum_{j=1}^{n} \Delta[M]_j = \sum_{j=1}^{n} (Y_j - \mu)^2,$$

(c) Since the process $[M]$ is integrable, we know there exists a unique increasing predictable and integrable process $\langle M\rangle = (\langle M\rangle_n)_{n\in\mathbb{N}_0}$ null at 0 such that $[M] - \langle M\rangle$ is a martingale. Thus, for each $n \in \mathbb{N}$, it holds that

$$E\Big[[M]_n - \langle M\rangle_n \,\Big|\, \mathcal{F}_{n-1}\Big] = [M]_{n-1} - \langle M\rangle_{n-1}.$$

The fact that $\langle M\rangle$ is predictable gives that

$$\begin{aligned}\langle M\rangle_n - \langle M\rangle_{n-1} &= E\left[[M]_n - [M]_{n-1} \,\middle|\, \mathcal{F}_{n-1}\right] \\ &= E\left[(Y_n - \mu)^2 \,\middle|\, \mathcal{F}_{n-1}\right] = \mathrm{Var}[Y_n] = \sigma^2,\end{aligned}$$

which in turn gives that $\langle M\rangle_n = n\sigma^2$.