Mathematical Foundations for Finance

Exercise Sheet 8

Please hand in your solutions by 12:00 on Wednesday, November 22 via the course homepage.

Exercise 8.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) For some constants $S_0 > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$, we define the *geometric Brownian motion* $S = (S_t)_{t\geqslant 0}$ as follows

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

Compute $\lim_{t\to\infty} S_t$ when $\mu \neq \frac{\sigma^2}{2}$. Determine whether the limit exists if $\mu = \frac{\sigma^2}{2}$.
Hint: You may use the law of the iterated logarithm.

(b) Prove that

$$E[W_t^3 - W_s^3 \mid \mathcal{F}_s] = 3(t-s)W_s \ P\text{-a.s., for } 0 \leq s < t.$$

Hint: You may compute $E[(W_t - W_s)^3 \mid \mathcal{F}_s]$.

Exercise 8.2 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. On this space, let M a local martingale null at 0 that satisfies $\sup_{0\leq t\leq T}|M_t| \in L^2$ for some $T \in \mathbb{R}$.

(a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: You may use dominated convergence theorem.

(b) Let $[M]$ be the square bracket process of M. Prove that

$$E\Big[[M]_t - [M]_s \,\Big|\, \mathcal{F}_s\Big] = \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s] \ P\text{-a.s., for } 0 \leq s \leq t \leq T.$$

Hint: You may use that $\mathrm{Var}[X \,|\, \mathcal{G}] = E\Big[(X - E\left[X \,|\, \mathcal{G}\right])^2 \,\Big|\, \mathcal{G}\Big]$.

Exercise 8.3 Let $(\Omega, \mathcal{F}, P)$ a probability space. We consider a sequence $(Y_k)_{k\in\mathbb{N}}$ of square-integrable and independent random variables and the filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \ldots, Y_k)$ for all $k \in \mathbb{N}$. We assume that $(Y_k)_{k\in\mathbb{N}}$

are identically distributed, with $\mu := E[Y_k] \in \mathbb{R}$ and $\sigma^2 := \mathrm{Var}[Y_k] > 0$, for $k \in \mathbb{N}$. Define the process $X = (X_n)_{n\in\mathbb{N}_0}$ by

$$X_n = \sum_{k=1}^{n} Y_k, \text{ for } n \in \mathbb{N}_0.$$

Note that X is adapted to $\mathbb{F}$ and integrable.

(a) Derive the Doob decomposition of X. In other words, find the martingale $M = (M_n)_{n\in\mathbb{N}_0}$ and the predictable and integrable process $A = (A_n)_{n\in\mathbb{N}_0}$ that are both null at zero and such that $X = M + A$ P-a.s. Deduce that M and A are square-integrable.
Hint: We have done it in Exercise 6.1(a).

(b) Find the optional quadratic variation $[M] = ([M]_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.
Hint: You may use Theorem V.1.1 in the lecture notes, and in particular the condition $\Delta[M] = (\Delta M)^2$.

(c) Explicitly derive the predictable quadratic variation $\langle M\rangle = (\langle M\rangle_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.