Exam simulation

Mathematical Foundations for Finance

Please hand in your solutions by 12:00 on Sunday, December 17 via the course homepage.

Question 1

For each of the following 12 subquestions, there is exactly one correct answer. For each correct answer you get 1 point, for each wrong answer you get $-1/2$ points, and for no answer you get 0 points. You get at least zero points for the whole exercise.

> Throughout subquestions (a) to (f), let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S})$ or shortly $(\widetilde{S}^0, \widetilde{S})$ be an undiscounted financial market in discrete time with a finite time horizon $T \in \mathbb{N}$. Assume that the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ is generated by $\widetilde{S}$, and $\widetilde{S}_0 := s_0 > 0$. Furthermore, suppose that $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$, where $r > -1$. The prices discounted by $\widetilde{S}^0$ are denoted by S^0 and S, respectively.

(a) Let $T = 1$. Suppose that $S = (S^1, S^2)$ evolves as

$$(S^1_0, S^2_0) = (6, 2), \quad (S^1_1, S^2_1) = \begin{cases} (9,3) & \text{with probability } p_1, \\ (6,2) & \text{with probability } p_2, \\ (3,1) & \text{with probability } p_3, \end{cases}$$

where $p_j > 0$ for $j = 1, 2, 3$ and $p_1 + p_2 + p_3 = 1$. Which of the following assertions is true?

(1) The market (S^0, S^1, S^2) is not arbitrage-free.
(2) The market (S^0, S^1, S^2) is arbitrage-free and complete.
(3) The market (S^0, S^1, S^2) is arbitrage-free and incomplete.

(b) Let $T = 1$. Suppose that $S_0 = 1$ and $S_1 = \exp(X)$, where X is normally distributed with mean $\mu = -0.2$ and variance $\sigma^2 = 0.5$. Which of the following assertions is true?

(1) S is a $(P, \mathbb{F})$-submartingale.
(2) S is a $(P, \mathbb{F})$-supermartingale.
(3) The market (S^0, S) is not arbitrage-free.

(c) Let $T = 1$. Assume that (S^0, S) is arbitrage-free and incomplete and let $\widetilde{C}$ denote an undiscounted payoff at time $T = 1$. Which of the following assertions is true?

(1) If $\widetilde{C}$ is attainable, then the market is complete.
(2) $\widetilde{C}$ is not attainable.
(3) C is attainable if and only if the undiscounted payoff $\widetilde{C} + \widetilde{S}_1$ is attainable.

(d) Assume that (S^0, S) is arbitrage-free and incomplete and let H^1 and H^2 be two discounted payoffs such that H^1 is attainable and H^2 is not attainable. Which of the following assertions is true?

(1) The discounted payoff $H^1 + H^2$ is never attainable.

(2) The discounted payoff $H^1 + H^2$ is always attainable.
(3) It cannot be said in general if the discounted payoff $H^1 + H^2$ is attainable or not.

(e) Let σ and τ be two $\mathbb{F}$-stopping times satisfying $\sigma \leq \tau$, P-a.s. Which of the following assertions is true?

(1) a random time $\widetilde{\sigma}$ satisfying $\sigma \leq \widetilde{\sigma} \leq \tau$, P-a.s., is an $\mathbb{F}$-stopping time.
(2) $\tau - \sigma$ is an $\mathbb{F}$-stopping time.
(3) $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

(f) Assume that $\widetilde{S}$ is a $(P, \mathbb{F})$-local submartingale. Which of the following assertions is true?

(1) The stopped process $(\widetilde{S}_{k \wedge \tau})_{k=0,1,\dots,T}$ is a $(P, \mathbb{F})$-submartingale for every $\mathbb{F}$-stopping time τ.
(2) The stochastic integral process $\varphi \bullet \widetilde{S}$ is a $(P, \mathbb{F})$-local submartingale for every bounded $\mathbb{F}$-predictable process φ.
(3) The process $\widetilde{S}$ is a $(P, \mathbb{F})$-submartingale if it is bounded from above.

Throughout subquestions (g) to (l), W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual hypotheses.

(g) Denote by $X = (X_t)_{t \geq 0}$ the stochastic exponential $\mathcal{E}(W) = (\mathcal{E}(W)_t)_{t \geq 0}$. Which of the following assertions is true?

(1) $X_t^2 = \exp(-2t)\mathcal{E}(2W)_t$ for any $t \geq 0$, P-a.s.
(2) $X_t^2 = \exp(t)\mathcal{E}(2W)_t$ for any $t \geq 0$, P-a.s.
(3) $X_t^2 = \exp(2t)\mathcal{E}(2W)_t$ for any $t \geq 0$, P-a.s.

(h) Which of the following processes is **not** a $(P, \mathbb{F})$-martingale?

(1) $(\exp(2t - 2W_t)_t)_{t \geq 0}$.
(2) $(\exp(-2t - 2W_t)_t)_{t \geq 0}$.
(3) $(\exp(-2t + 2W_t)_t)_{t \geq 0}$.

(i) Which of the following assertions is true?

(1) A $(P, \mathbb{F})$-local martingale is always a $(P, \mathbb{F})$-supermartingale.
(2) A $(P, \mathbb{F})$-local martingale is always a $(P, \mathbb{F})$-submartingale.
(3) A $(P, \mathbb{F})$-supermartingale which is also a $(P, \mathbb{F})$-submartingale is always a $(P, \mathbb{F})$-local martingale.

(j) Let M be a $(P, \mathbb{F})$-locally square-integrable local martingale null at 0. Which of the following assertions is true?

(1) If H is $\mathbb{F}$-predictable, bounded and positive, then $H \bullet M$ is a $(P, \mathbb{F})$-supermartingale.
(2) If $|M_t| < \infty$, P-a.s., for all $t > 0$, then M is a $(P, \mathbb{F})$-martingale.
(3) If $|M_t| \leq L$, P-a.s., for all $t \geq 0$, for some P-integrable random variable L, then M is a $(P, \mathbb{F})$-martingale.

(k) Which of the following statements is **not** true?

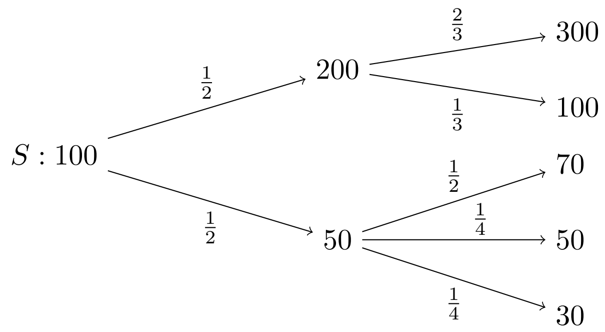
(1) The sample paths of X with $X_t := \int_0^t W_s \mathrm{d}s$, for $t \geq 0$, are P-a.s. of finite variation.
(2) The sample paths of $[M]$ are P-a.s. of finite variation for any $(P, \mathbb{F})$-martingale M.
(3) The sample paths of Y with $Y_t := \int_0^t s \mathrm{d}W_s$, for $t \geq 0$, are P-a.s. of finite variation.

(l) Let M be a $(P, \mathbb{F})$-local martingale and A be a process whose sample paths are P-a.s. of finite variation. Which of the following statements is true?

(1) $H \bullet M$ is a $(P, \mathbb{F})$-martingale, if H is bounded and $\mathbb{F}$-predictable.
(2) $H \bullet A$ has finite variation, if H is bounded and $\mathbb{F}$-predictable.
(3) $H \bullet M$ is a $(P, \mathbb{F})$-martingale, if M is a $(P, \mathbb{F})$-martingale and H is bounded and $\mathbb{F}$-predictable.

Question 2

Consider a financial market $(\widetilde{S}^0, \widetilde{S})$ consisting of a bank account and one stock. The movements of the discounted stock price S are described by the following tree, where the numbers beside the branches denote transition probabilities.



More precisely, let $(\Omega, \mathcal{F}, P)$ be the probability space with $\Omega := \{(1,1),(1,2),(1,3),(2,1),(2,2)\}$, $\mathcal{F} = 2^\Omega$ and the probability measure P defined by $P[\{(x_1, x_2)\}] := p_{x_1} p_{x_1,x_2}$, where

$$p_1 = p_2 = \frac{1}{2} \text{ and } p_{1,1} = p_{1,2} = \frac{1}{4}, \quad p_{1,3} = \frac{1}{2}, \quad p_{2,1} = \frac{1}{3}, \quad p_{2,2} = \frac{2}{3}.$$

The discounted bank account process $(S_k^0)_{k=0,1,2}$ is given by $S_k^0 = 1$, for $k = 0, 1, 2$, and the discounted stock price process $(S_k)_{k=0,1,2}$ by

$$S_0 = 100 \qquad S_1((1,j)) = 50, \quad S_1((2,\ell)) = 200, \quad j = 1,2,3, \ \ell = 1,2.$$
$$S_2((1,1)) = 30, \quad S_2((1,2)) = 50, \quad S_2((1,3)) = 70, \quad S_2((2,1)) = 100, \quad S_2((2,2)) = 300.$$

The filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ is given by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(S_1)$ and $\mathcal{F}_2 := \sigma(S_1, S_2) = \mathcal{F}$.

(a) Show that the market is free of arbitrage and explicitly describe the set $\mathbb{P}_e(S)$ of all equivalent martingale measures for S.

(b) For $K \in (30, 300)$, let $C^K = (S_2 - K)^+$ denote the discounted payoff of an European call option on S with strike K and maturity $T = 2$. For which values of K is this option attainable? Justify your answer.

(c) Assume that the option C^{80} is traded at a price of 50 at time 0. Show that the extended market $(S^0, S, (50, C^{80}))$ admits arbitrage.
Hint: Argue whether one should buy the option or sell it short.

Question 3

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$, where $T \in \mathbb{N}$. We set $\mathcal{F}_0 := \{\emptyset, \Omega\}$, and assume that $\mathcal{F}_T = \mathcal{F}$. Let $\mathcal{D} > 0$ be a random variable on $(\Omega, \mathcal{F})$ with $E^P[\mathcal{D}] = 1$. We introduce a probability measure Q on $(\Omega, \mathcal{F})$ such that

$$Q[A] := E^P[\mathcal{D}\mathbf{1}_A], \text{ for } A \in \mathcal{F}.$$

We define the process $Z = (Z_k)_{k=0,1,\dots,T}$ by $Z_k := E^P[\mathcal{D}|\mathcal{F}_k]$.

(a) Let X_n be a simple random variable of the form

$$X_n := \sum_{i=1}^{n} x_{i,n}\mathbf{1}_{A_{i,n}} \tag{1}$$

for some constants $x_{1,n}, \dots, x_{n,n} \geq 0$ and some sets $A_{1,n}, \dots, A_{n,n} \in \mathcal{F}$. Show that $E^Q[X_n] = E^P[\mathcal{D}X_n]$.

(b) Show that $E^Q[X] = E^P[\mathcal{D}X]$ for any random variable $X \geq 0$.
Hint: Without giving a proof, assume that there exists a nondecreasing sequence $(X_n)_{n\in\mathbb{N}}$ of simple random variables described by (1) such that $\lim_{n\to\infty} X_n = X$, P-a.s.

(c) Infer from (b) that $E^Q[Y] = E^P[Z_k Y]$ for any $\mathcal{F}_k$-measurable random variable $Y \geq 0$.

Question 4

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space which satisfies the usual conditions. Fix some $\alpha > 0$. Assume that W is a $(P, \mathbb{F})$-Brownian motion, and X is a $(P, \mathbb{F})$-semimartingale satisfying

$$\begin{cases} \mathrm{d}X_t = \alpha X_t \mathrm{d}t + \mathrm{d}W_t, \ P\text{-a.s.} \\ X_0 = 0, \ P\text{-a.s.} \end{cases}$$

(a) Show that $(e^{-\alpha t}X_t)_{t\geq 0}$ is a $(P, \mathbb{F})$-local martingale. Write X as a stochastic integral.

(b) Fix some $T > 0$. Prove that $(e^{-\alpha t}X_t)_{t\in[0,T]}$ is a $(P, \mathbb{F})$-martingale in $\mathcal{M}_0^2$. Deduce $E[X_T]$ and $Var(X_T)$.

(c) Define the process $Z = (Z_t)_{t\geq 0}$, where

$$Z_t := \exp\bigg\{ -\alpha \int_0^t X_s \mathrm{d}W_s - \frac{\alpha^2}{2}\int_0^t X_s^2 \mathrm{d}s \bigg\}.$$

Prove that Z is a $(P, \mathbb{F})$-local martingale.

Question 5

Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W satisfying the usual conditions. Consider the Black–Scholes model with time-dependent interest rate r, i.e., the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S} = (\widetilde{S}_t)_{t\in[0,T]}$ satisfy the SDEs

$$\frac{\mathrm{d}\widetilde{S}_t}{\widetilde{S}_t} = \mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}W_t, \quad \widetilde{S}_0 = 1,\ P\text{-a.s.}$$
$$\frac{\mathrm{d}\widetilde{S}^0_t}{\widetilde{S}^0_t} = r(t)\mathrm{d}t, \quad \widetilde{S}^0_0 = 1,\ P\text{-a.s.}$$

Here, $\mu_1 \in \mathbb{R}$ and $\sigma_1 > 0$. The discount rate $r : [0,T] \to \mathbb{R}$ is of the form $r(t) = \sum_{i=0}^{n-1} d_i \mathbf{1}_{(t_i,t_{i+1}]}(t)$, for some constants $d_i \in \mathbb{R}$ and a partition $0 = t_0 < \cdots < t_n = T$.

(a) Let $\lambda : [0,T] \to \mathbb{R}$ be in $L^2_{loc}(W)$. Show that the process $Z = (Z_t)_{t\in[0,T]}$ defined by

$$Z_t := \mathcal{E}\left(\int_0^\cdot \lambda(s)\mathrm{d}W_s\right)_t$$

is a $(P, \mathbb{F})$-martingale.

(b) Find the density process Z of a probability measure $Q \approx P$ on $\mathcal{F}_T$ such that the discounted price process $S := \widetilde{S}/\widetilde{S}^0$ is a $(Q, \mathbb{F})$-martingale. Compute the dynamics of S under the measure Q.

(c) Hedge the discounted power option for $p > 0$, i.e., find (V_0, ϑ) such that

$$H := \frac{(\widetilde{S}_T)^p}{\widetilde{S}^0_T} = V_0 + \int_0^T \vartheta_t \mathrm{d}S_t,\ P\text{-a.s.}$$