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Exam simulation

Mathematical Foundations for Finance

Solutions

Solution 1

The correct answers are:

(a) (3)

(b) (1)

(c) (3)

(d) (1)

(e) (3)

(f) (3)

(g) (2)

(h) (1)

(i) (3)

(j) (3)

(k) (3)

(l) (2)

Solution 2

(a) Any probability measure Q equivalent to P on $\mathcal{F}_2$ can be described by

$$Q[\{(x_1, x_2)\}] := q_{x_1} q_{x_1,x_2},$$

where $q_{x_1}, q_{x_1,x_2} \in (0,1)$ and $\sum_{x_1 \in \{1,2\}} q_{x_1} = 1$, $\sum_{x_2 \in \{1,2,3\}} q_{1,x_2} = 1$, $\sum_{x_2 \in \{1,2\}} q_{2,x_2} = 1$. Next, since $\mathcal{F}_0$ is trivial and $\mathcal{F}_1 = \sigma(S_1^1)$ (and S_1^1 only takes two values), S^1 is a Q-martingale if and only if

$$\mathbb{E}^Q[S_1] = 100, \quad \mathbb{E}^Q[S_2 \,|\, S_1 = 200] = 200 \quad \text{and} \quad \mathbb{E}^Q[S_2 \,|\, S_1 = 50] = 50.$$

Thus, $q_1, q_2, q_{1,1}, q_{1,2}, q_{1,3}, q_{2,1}, q_{2,2} \in (0,1)$ define an equivalent martingale measure for S if and only if they satisfy the three systems of equations

$$\begin{cases} q_1 + q_2 &= 1, \\ 50q_1 + 200q_2 &= 100; \end{cases} \tag{I}$$

$$\begin{cases} q_{2,1} + q_{2,2} &= 1, \\ 300q_{2,1} + 100q_{2,2} &= 200; \end{cases} \tag{II}$$

$$\begin{cases} q_{1,1} + q_{1,2} + q_{1,3} &= 1, \\ 30q_{1,1} + 50q_{1,2} + 70q_{1,3} &= 50. \end{cases} \tag{III}$$

It is straightforward to check that the solution to (I) and (II) are given by

$$q_1 = \frac{2}{3}, \quad q_2 = \frac{1}{3} \quad \text{and} \quad q_{2,1} = \frac{1}{2}, \quad q_{2,2} = \frac{1}{2}.$$

Moreover, (III) is equivalent to

$$\begin{cases} q_{1,1} + q_{1,2} + q_{1,3} &= 1, \\ -q_{1,1} + q_{1,3} &= 0. \end{cases} \tag{III'}$$

Recalling that $q_{1,1}, q_{1,2}, q_{1,3} \in (0,1)$ shows that the solution to (III') is given by

$$q_{1,1} = \rho, \quad q_{1,2} = 1 - 2\rho, \quad q_{1,3} = \rho, \text{ where } \rho \in (0, 1/2).$$

Thus, $\mathbb{P}_e(S^1) = \{Q^\rho : \rho \in (0,1/2)\}$, where $Q^\rho[\{(x_1,x_2)\}] = q^\rho_{x_1} q^\rho_{x_1,x_2}$ with

$$q^\rho_1 = \frac{2}{3}, \quad q^\rho_2 = \frac{1}{3}, \quad q^\rho_{1,1} = \rho, \quad q^\rho_{1,2} = 1 - 2\rho, \quad q^\rho_{1,3} = \rho \quad \text{and} \quad q^\rho_{2,1} = \frac{1}{2}, \quad q^\rho_{2,2} = \frac{1}{2}.$$

(b) The call option is attainable for $K \in [70, 300)$ and **not** attainable for $K \in (30, 70]$. Indeed, fix $\rho \in (0, 1/2)$. Then,

$$\mathbb{E}^{Q^\rho}\left[C^K\right] = \begin{cases} \frac{1}{3} \times \frac{1}{2}(300 - K) = \frac{300-K}{6}, & K \in [100, 300), \\ \frac{300-K}{3} + \frac{1}{3} \times \frac{1}{2}(100 - K) = \frac{200-K}{3}, & K \in [70, 100), \\ \frac{200-K}{3} + \frac{2}{3}\rho(70 - K) = \frac{200-K}{3} + \rho\frac{140-2K}{3}, & K \in [50, 70), \\ \frac{200-K}{3} + \rho\frac{140-2K}{3} + \frac{2}{3}(1 - 2\rho)(50 - K) = 100 - K + \rho\frac{2K-60}{3}, & K \in [30, 50). \end{cases}$$

For $K \in [70, 300)$, the mapping $(0, 1/2) \ni \rho \mapsto \mathbb{E}^{Q^\rho}[C^K]$ is constant, and for $K \in [30, 70)$ this mapping is non-constant. The claim follows from Theorem III.1.2 in the lecture notes.

(c) It follows from the solution of part (b) that the option C^{80} is attainable and that its arbitrage free price is given by $V_0^{C^{80}} = 40$. Thus, the traded price of 50 at time 0 is too high. Therefore, we sell the option short and use the money to replicate it and put the rest in the bank account, i.e., we choose $\varphi = (0, \vartheta)$, where ϑ is such that $40 + G_2(\vartheta) = C^{80}$. With this choice, we have

$$G_2(\vartheta) - (C^{80} - 50) = C^{80} - 40 - C^{80} + 50 = 10 > 0.$$

Thus, this is an arbitrage opportunity.

Solution 3

(a) Let X_n be a simple random variable of the form

$$X_n = \sum_{i=1}^{n} x_{i,n} \mathbf{1}_{A_{i,n}}$$

for some constants $x_{1,n}, \ldots, x_{n,n} \geq 0$ and some sets $A_{1,n}, \ldots, A_{n,n} \in \mathcal{F}$. We compute that

$$\begin{aligned} E^Q[X_n] &= \sum_{i=1}^{n} x_{i,n} E^Q[\mathbf{1}_{A_{i,n}}] = \sum_{i=1}^{n} x_{i,n} Q[A_{i,n}] = \sum_{i=1}^{n} x_{i,n} E^P[\mathcal{D}\mathbf{1}_{A_{i,n}}] \\ &= E^P\left[\mathcal{D} \sum_{i=1}^{n} x_{i,n} \mathbf{1}_{A_{i,n}}\right] = E^P[\mathcal{D} X_n]. \end{aligned}$$

(b) Let $(X_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of simple random variables such that $\lim_{n\to\infty} X_n = X$, P-a.s. By the monotone convergence theorem, we immediately obtain that $E^Q[X_n] \uparrow E^Q[X]$, as $n \to \infty$. But since $\mathcal{D} > 0$, we also clearly have that $\mathcal{D}X_n \uparrow \mathcal{D}X$, P-a.s., and another application of the monotone convergence theorem thus gives that $E^P[\mathcal{D}X_n] \uparrow E^P[\mathcal{D}X]$. Therefore, taking the limit on both sides of the result in (a) gives $E^Q[X] = E^P[\mathcal{D}X]$ as desired.

(c) We compute

$$E^Q[Y] = E^P[\mathcal{D}Y] = E^P\left[E^P[\mathcal{D}Y|\mathcal{F}_k]\right] = E^P\left[Y E^P[\mathcal{D}|\mathcal{F}_k]\right] = E^P[Z_k Y].$$

The first equality uses the result in (b), the second one uses the tower property of the conditional expectation, the third one the $\mathcal{F}_k$-measurability and the nonnegativity of Y, and the last one the definition of Z_k.

Solution 4

(a) By applying Itô's formula, we get

$$\mathrm{d}(e^{-\alpha t}X_t) = -\alpha e^{-\alpha t}X_t\mathrm{d}t + e^{-\alpha t}(\alpha_t X_t\mathrm{d}t + \mathrm{d}W_t) = e^{-\alpha t}\mathrm{d}W_t,\ P\text{-a.s.}$$

Or, equivalently,

$$e^{-\alpha t}X_t = \int_0^t e^{-\alpha s}\mathrm{d}W_s,\ P\text{-a.s., for any } t \geq 0, \tag{1}$$

where we have used the fact that $X_0 = 0$, P-a.s. We conclude that $(e^{-\alpha t}X_t)_{t\geq 0}$ is a $(P,\mathbb{F})$-local martingale. Moreover, from (1) we get that

$$X_t = e^{\alpha t}\int_0^t e^{-\alpha s}\mathrm{d}W_s = \int_0^t e^{\alpha(t-s)}\mathrm{d}W_s,\ P\text{-a.s., for any } t \geq 0.$$

(b) Let us fix some $T > 0$. It holds that

$$E\left[\int_0^T e^{-2\alpha s}\mathrm{d}[W]_s\right] = E\left[\int_0^T e^{-2\alpha s}\mathrm{d}s\right] = \frac{1-e^{2\alpha T}}{2\alpha} < \infty.$$

The equation (1) implies that $(e^{-\alpha t}X_t)_{t\in[0,T]}$ is a $(P,\mathbb{F})$-martingale in $\mathcal{M}_0^2$. Moreover, from the martingale property, we deduce that

$$E[X_T] = E\left[\int_0^T e^{\alpha(T-s)}\mathrm{d}W_s\right] = e^{\alpha T}E\left[\int_0^T e^{-\alpha s}\mathrm{d}W_s\right] = 0.$$

Itô's isometry implies that

$$E\big[X_T^2\big] = E\left[\left(\int_0^T e^{\alpha(T-s)}\mathrm{d}W_s\right)^2\right] = E\left[\int_0^T e^{2\alpha(t-s)}\mathrm{d}s\right] = \frac{e^{2\alpha T}-1}{2\alpha}.$$

We can conclude that $Var(X_T) = E[X_T^2]$.

(c) It is straightforward to verify that $Z_t = \mathcal{E}(-\alpha\int_0^\cdot X_s\mathrm{d}W_s)_t$ for any $t \geq 0$, P-a.s. It follows that

$$\mathrm{d}Z_t = Z_t\big(-\alpha X_t\mathrm{d}W_t\big),\ P\text{-a.s.}$$

We can conclude that Z is a $(P,\mathbb{F})$-local martingale.

Solution 5

(a) Let us define the process $Y = (Y_t)_{t\in[0,T]}$ by

$$Y_t := \int_0^t \lambda(s)\mathrm{d}W_s.$$

By the assumption that $\lambda \in L^2_{loc}(W)$, we see that Y is a continuous $(P, \mathbb{F})$-local martingale. Moreover, we could recall the Novikov's condition that says that if $Y = (Y_t)_{t\in[0,T]}$ is a continuous $(P, \mathbb{F})$-local martingale with $Y_0 = 0$ and $E\left[e^{\frac{1}{2}\langle Y\rangle_T}\right] < \infty$, then $\mathcal{E}(Y)$ is a true $(P, \mathbb{F})$-martingale on $[0, T]$. In our case, we have that

$$E\left[e^{\frac{1}{2}\langle Y\rangle_T}\right] = E\left[e^{\frac{1}{2}\int_0^T \lambda^2(s)ds}\right] = e^{\frac{1}{2}\int_0^T \lambda^2(s)ds} < \infty,$$

so the result follows.

(b) We first compute the P-dynamics of the discounted price process $S = \widetilde{S}/\widetilde{S}^0$. Direct application of Itô's formula with the C^2 function $\mathbb{R}^2_{++} \ni (x, y) \mapsto x/y$ yields

$$\frac{\mathrm{d}S_t}{S_t} = \big(\mu_1 - r(t)\big)\mathrm{d}t + \sigma_1\mathrm{d}W_t,\ P\text{-a.s.} \tag{2}$$

Define $\lambda : [0, T] \to \mathbb{R}$ by $\lambda(s) := (\mu_1 - r(s))/\sigma_1$ and the process $Z = (Z_t)_{t\in[0,T]}$ by

$$Z_t := \mathcal{E}\left(-\int_0^\cdot \lambda(s)\mathrm{d}W_s\right)_t.$$

Since λ is left-continuous and bounded, (a) gives that Z is a positive $(P, \mathbb{F})$-martingale with $E[Z_t] = 1$ for all $t \in [0, T]$, and thus the density process of some measure $Q \approx P$. Girsanov's theorem gives that

$$W_t^Q := W_t - \left\langle W, -\int_0^\cdot \lambda(s)\mathrm{d}W_s\right\rangle_t = W_t + \int_0^t \lambda(s)\mathrm{d}s = W_t + \int_0^t \frac{\mu_1 - r(s)}{\sigma_1}\mathrm{d}s,\ P\text{-a.s.},$$

is a $(Q, \mathbb{F})$-Brownian motion. By (2), the Q-dynamics of S is given by

$$\frac{\mathrm{d}S_t}{S_t} = \big(\mu_1 - r(t)\big)\mathrm{d}t - \big(\mu_1 - r(t)\big)\mathrm{d}t + \sigma_1\mathrm{d}\left(W_t + \int_0^t \frac{\mu_1 - r(s)}{\sigma_1}\mathrm{d}s\right) = \sigma_1\mathrm{d}W_t^Q,\ P\text{-a.s.}$$

In other words, $S = \mathcal{E}(\sigma_1 W^Q)$, which is a $(Q, \mathbb{F})$-martingale.

(c) The arbitrage-free price at time t of the discounted payoff H is given by

$$\begin{aligned}
V_t &= E^Q\left[\frac{(\widetilde{S}_T)^p}{\widetilde{S}_T^0}\middle|\mathcal{F}_t\right] = \left(\widetilde{S}_T^0\right)^{p-1} E^Q\left[(S_T^1)^p\middle|\mathcal{F}_t\right] = \left(\widetilde{S}_T^0\right)^{p-1}(S_t)^p E^Q\left[\left(\frac{S_T^1}{S_t^1}\right)^p\middle|\mathcal{F}_t\right] \\
&= e^{(p-1)\int_0^T r(s)\mathrm{d}s}(S_t)^p E^Q\left[e^{p\sigma_1(W_T^Q - W_t^Q) - p\frac{\sigma_1^2}{2}(T-t)}\middle|\mathcal{F}_t\right] \\
&= e^{(p-1)\int_0^T r(s)\mathrm{d}s - p\frac{\sigma_1^2}{2}(T-t)}(S_t)^p E^Q\left[e^{p\sigma_1(W_T^Q - W_t^Q)}\right] \\
&= e^{(p-1)\int_0^T r(s)\mathrm{d}s - p\frac{\sigma_1^2}{2}(T-t)}(S_t)^p e^{\frac{p^2\sigma_1^2(T-t)}{2}} \\
&=: v(t, S_t),
\end{aligned}$$

where we have used that the increment $W_T^Q - W_t^Q$ is independent of $\mathcal{F}_t$ under Q and normally distributed with mean 0 and variance $T - t$. Consequently, the delta of H is given by

$$\vartheta_t = \left.\frac{\partial v}{\partial x}\right|_{(t,x)=(t,S_t)} = pe^{(p-1)\int_0^T r(s)\mathrm{d}s - p\frac{\sigma_1^2}{2}(T-t)}(S_t)^{p-1}e^{\frac{p^2\sigma_1^2(T-t)}{2}},$$

and the price at time 0 is

$$V_0 = v(0, S_0) = e^{(p-1)\int_0^T r(s)\mathrm{d}s - p\frac{\sigma_1^2}{2}T}e^{\frac{p^2\sigma_1^2 T}{2}}.$$