Non-Life Insurance: Mathematics and Statistics

Exercise sheet 1

Exercise 1.1 Discrete Distribution
Suppose that N follows a geometric distribution with parameter $p \in (0,1)$, i.e.

$$\mathbb{P}[N = k] = \begin{cases} (1-p)^{k-1}p, & \text{if } k \in \mathbb{N}_{>0}, \\ 0, & \text{else.} \end{cases}$$

(a) Show that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $n \in \mathbb{N}_{>0}$. Calculate $\mathbb{P}[N \geq n]$.

(c) Calculate $\mathbb{E}[N]$.

(d) Let $r < -\log(1-p)$. Calculate the moment generating function $M_N(r) = \mathbb{E}[\exp\{rN\}]$ of N.

(e) Calculate $\frac{d}{dr} M_N(r)\big|_{r=0}$. What do you observe?

Exercise 1.2 Absolutely Continuous Distribution
Suppose that Y follows an exponential distribution with parameter $\lambda > 0$, i.e. the density f_Y of Y is given by

$$f_Y(x) = \begin{cases} \lambda \exp\{-\lambda x\}, & \text{if } x \geq 0, \\ 0, & \text{else.} \end{cases}$$

(a) Show that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $0 < y_1 < y_2$. Calculate $\mathbb{P}[y_1 \leq Y \leq y_2]$.

(c) Calculate $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

(d) Let $r < \lambda$. Calculate the cumulant generating function $\log M_Y(r) = \log \mathbb{E}[\exp\{rY\}]$ of Y.

(e) Calculate $\frac{d^2}{dr^2} \log M_Y(r)\big|_{r=0}$. What do you observe?

Exercise 1.3 Gaussian Distribution
For a random variable X we write $X \sim \mathcal{N}(\mu, \sigma^2)$ if X follows a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. The density f_X of $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$f_X(x) \;=\; \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2} \right\}, \qquad \text{for all } x \in \mathbb{R}.$$

(a) Show that the moment generating function M_X of $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$M_X(r) \;=\; \exp\left\{ r\mu + \frac{r^2\sigma^2}{2} \right\}, \qquad \text{for all } r \in \mathbb{R}.$$

(b) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $a, b \in \mathbb{R}$. Show that

$$a + bX \;\sim\; \mathcal{N}(a + b\mu, b^2\sigma^2).$$

(c) Let $X_1, \ldots, X_n$ be independent with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i \in \{1, \ldots, n\}$. Show that

$$\sum_{i=1}^n X_i \;\sim\; \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2 \right).$$

Exercise 1.4 χ^2-Distribution
For all $k \in \mathbb{N}_{>0}$ we assume that X_k has a χ^2-distribution with k degrees of freedom, i.e. X_k has density

$$f_{X_k}(x) \;=\; \begin{cases} \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} \exp\{-x/2\}, & \text{if } x \geq 0, \\ 0, & \text{else.} \end{cases}$$

(a) Let M_{X_k} be the moment generating function of X_k. Show that

$$M_{X_k}(r) \;=\; \frac{1}{(1-2r)^{k/2}}, \qquad \text{for } r < 1/2.$$

(b) Let $Z \sim \mathcal{N}(0,1)$. Show that $Z^2 \overset{(d)}{=} X_1$.

(c) Let $Z_1, \ldots, Z_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. Show that $\sum_{i=1}^k Z_i^2 \overset{(d)}{=} X_k$.