

# Non-Life Insurance: Mathematics and Statistics

## Exercise sheet 12

### Exercise 12.1 (Inhomogeneous) Credibility Estimators for Claim Counts

Suppose that in Table 1, we are given the current year's claim counts data for 5 different regions, where, for all  $i \in \{1, \dots, 5\}$ ,  $v_{i,1}$  denotes the number of policies in region  $i$  and  $N_{i,1}$  the number of claims in region  $i$ . We assume that we are in the Bühlmann-Straub model framework with  $I = 5$ ,  $T = 1$  and

$$N_{i,1} | \Theta_i \sim \text{Poi}(\mu(\Theta_i)v_{i,1}),$$

with  $\mu(\Theta_i) = \Theta_i \lambda_0$  and  $\lambda_0 = 0.088$ , for all  $i \in \{1, \dots, 5\}$ . Moreover, we assume that the pairs  $(\Theta_1, N_{1,1}), \dots, (\Theta_5, N_{5,1})$  are independent and  $\Theta_1, \dots, \Theta_5$  are i.i.d. with  $\mathbb{E}[\Theta_1] = 1$  and  $\Theta_1 > 0$  a.s. Finally, we set  $\tau^2 = \text{Var}(\mu(\Theta_1)) = 0.00024$ .

region $i$	$v_{i,1}$	$N_{i,1}$
1	50'061	3'880
2	10'135	794
3	121'310	8'941
4	35'045	3'448
5	4'192	314

Table 1: Observed numbers of policies  $v_{i,1}$  and numbers of claims  $N_{i,1}$  in the 5 regions.

- (a) Calculate the inhomogeneous credibility estimator  $\widehat{\mu(\Theta_i)}$  for each region  $i \in \{1, \dots, 5\}$  and comment on the results. What would we observe if we decreased the volatility  $\tau^2$  between the risk classes?
- (b) We denote next year's numbers of policies by  $v_{1,2}, \dots, v_{5,2}$  and next year's numbers of claims by  $N_{1,2}, \dots, N_{5,2}$ . Suppose that  $N_{i,1}$  and  $N_{i,2}$  are independent, conditionally given  $\Theta_i$ , for all  $i \in \{1, \dots, 5\}$ , and that the number of policies grows 5% in each region. For all  $i \in \{1, \dots, 5\}$ , under the assumption  $N_{i,2} | \Theta_i \sim \text{Poi}(\mu(\Theta_i)v_{i,2})$ , calculate the mean square error of prediction

$$\mathbb{E} \left[ \left( \frac{N_{i,2}}{v_{i,2}} - \widehat{\mu(\Theta_i)} \right)^2 \right].$$

### Exercise 12.2 (Homogeneous) Credibility Estimators for Claim Sizes

Suppose that in Table 2, we are given claim size data for two different years and four different risk classes, where  $v_{i,t}$  denotes the number of claims in risk class  $i$  and year  $t$  and  $Y_{i,t}$  the total claim size in risk class  $i$  and year  $t$ , for all  $i \in \{1, 2, 3, 4\}$  and  $t \in \{1, 2\}$ . We assume that we are in the Bühlmann-Straub model framework with  $I = 4$ ,  $T = 2$  and

$$Y_{i,t} | \Theta_i \sim \Gamma(\mu(\Theta_i)cv_{i,t}, c),$$

with  $\mu(\Theta_i) = \Theta_i$  and  $c > 0$ , for all  $i \in \{1, 2, 3, 4\}$  and  $t \in \{1, 2\}$ . Moreover, we assume that  $(\Theta_1, Y_{1,1}, Y_{1,2}), \dots, (\Theta_4, Y_{4,1}, Y_{4,2})$  are independent and that  $\Theta_1, \dots, \Theta_4$  are i.i.d. with  $\mathbb{E}[\Theta_1^2] < \infty$  and  $\Theta_1 > 0$  a.s. Finally, we also assume that  $Y_{i,1}$  and  $Y_{i,2}$  are independent, conditionally given  $\Theta_i$ , for all  $i \in \{1, 2, 3, 4\}$ .

risk class $i$	$v_{i,1}$	$Y_{i,1}$	$v_{i,2}$	$Y_{i,2}$
1	1'058	8'885'738	1'111	13'872'665
2	3'146	7'902'445	3'303	4'397'183
3	238	2'959'517	250	6'007'351
4	434	10'355'286	456	15'629'998

Table 2: Observed numbers of claims  $v_{i,1}$  and  $v_{i,2}$  and total claim sizes  $Y_{i,1}$  and  $Y_{i,2}$  in the 4 risk classes.

- (a) Calculate the homogeneous credibility estimator  $\widehat{\mu(\Theta_i)}^{\text{hom}}$  for each risk class  $i \in \{1, 2, 3, 4\}$  and comment on the results.
- (b) We denote next year's numbers of claims by  $v_{1,3}, \dots, v_{4,3}$  and next year's total claim sizes by  $Y_{1,3}, \dots, Y_{4,3}$ . Suppose that  $Y_{i,1}, Y_{i,2}$  and  $Y_{i,3}$  are independent, conditionally given  $\Theta_i$ , for all  $i \in \{1, 2, 3, 4\}$ , and that the number of claims grows 5% in each risk cell. For all  $i \in \{1, 2, 3, 4\}$ , under the assumption  $Y_{i,3}|\Theta_i \sim \Gamma(\mu(\Theta_i)cv_{i,3}, c)$ , estimate the mean square error of prediction

$$\mathbb{E} \left[ \left( \frac{Y_{i,3}}{v_{i,3}} - \widehat{\mu(\Theta_i)}^{\text{hom}} \right)^2 \right].$$

**Exercise 12.3 Degenerate MLE and the Poisson-Gamma Model**

Suppose that in a given line of business, we observed the following claim counts data for  $T = 5$  years  $t = 1, \dots, T$ :

$t$	1	2	3	4	5
$N_t$	0	0	0	0	0
$v_t$	10	10	10	10	10

Table 3: Observed claim counts  $N_t$  and corresponding volumes  $v_t$  for  $T = 5$  years  $t = 1, \dots, T$ .

- (a) First, we assume a Poisson model for the claim counts, i.e.  $N_1, \dots, N_T$  are independent with  $N_t \sim \text{Poi}(\lambda v_t)$ ,  $t = 1, \dots, T$ , for an unknown claim frequency parameter  $\lambda > 0$ . Calculate the MLE  $\widehat{\lambda}_T$  of  $\lambda$ . Does this estimate  $\widehat{\lambda}_T$  make sense for premium calculation?
- (b) Now we assume a Poisson-gamma model for the claim counts, i.e.  $\Lambda \sim \Gamma(\gamma, c)$  with  $\gamma = 1$  and  $c = 50$ , and, conditionally given  $\Lambda$ ,  $N_1, \dots, N_T$  are independent with  $N_t \sim \text{Poi}(\Lambda v_t)$ ,  $t = 1, \dots, T$ .
- (i) Determine the prior estimator  $\lambda_0$  and the posterior estimator  $\widehat{\lambda}_T^{\text{post}}$ , conditionally given data  $(N_1, v_1), \dots, (N_T, v_T)$ , of the unknown parameter  $\Lambda$ .
- (ii) Find the credibility weight  $\alpha_T \in (0, 1)$  such that

$$\widehat{\lambda}_T^{\text{post}} = \alpha_T \widehat{\lambda}_T + (1 - \alpha_T) \lambda_0.$$

- (iii) Suppose we have an additional observation  $(N_{T+1}, v_{T+1}) = (1, 10)$  within the Poisson-gamma model framework and that  $\widehat{\lambda}_{T+1}^{\text{post}}$  denotes the posterior estimator, conditionally given data  $(N_1, v_1), \dots, (N_{T+1}, v_{T+1})$ , of the unknown parameter  $\Lambda$ . Find the credibility weight  $\beta_{T+1} \in (0, 1)$  such that

$$\widehat{\lambda}_{T+1}^{\text{post}} = \beta_{T+1} \frac{N_{T+1}}{v_{T+1}} + (1 - \beta_{T+1}) \widehat{\lambda}_T^{\text{post}}.$$

- (c) Finally, we assume a Poisson-normal model for the claim counts, i.e.  $\Lambda \sim \mathcal{N}(\mu, \sigma^2)$  with mean  $\mu \in \mathbb{R}$  and variance  $\sigma^2 > 0$ , and, conditionally given  $\Lambda$ ,  $N_1, \dots, N_T$  are independent with  $N_t \sim \text{Poi}(\Lambda v_t)$ ,  $t = 1, \dots, T$ . Is such a model reasonable?

**Exercise 12.4 Pareto-Gamma Model**

Suppose that  $\Lambda \sim \Gamma(\gamma, c)$  with prior shape parameter  $\gamma > 0$  and prior scale parameter  $c > 0$  and, conditionally given  $\Lambda$ , the components of  $\mathbf{Y} = (Y_1, \dots, Y_T)$  are independent with  $Y_t \sim \text{Pareto}(\theta, \Lambda)$  for some threshold  $\theta > 0$ , for all  $t \in \{1, \dots, T\}$ .

- (a) Show that the posterior distribution of  $\Lambda$ , conditional on  $\mathbf{Y}$ , is given by

$$\Lambda | \mathbf{Y} \sim \Gamma \left( \gamma + T, c + \sum_{t=1}^T \log \frac{Y_t}{\theta} \right).$$

- (b) For the estimation of the unknown tail index parameter  $\Lambda$  of the Pareto distributions, we define the prior estimator  $\lambda_0 = \mathbb{E}[\Lambda]$  and the observation based estimator (MLE of the Pareto tail index parameter)

$$\hat{\lambda}_T = \frac{T}{\sum_{t=1}^T \log \frac{Y_t}{\theta}}.$$

Find the credibility weight  $\alpha_T \in (0, 1)$  such that the posterior estimator  $\hat{\lambda}_T^{\text{post}} = \mathbb{E}[\Lambda | \mathbf{Y}]$  has the credibility form

$$\hat{\lambda}_T^{\text{post}} = \alpha_T \hat{\lambda}_T + (1 - \alpha_T) \lambda_0.$$

- (c) Show that for the (conditional mean square error) uncertainty of the posterior estimator  $\hat{\lambda}_T^{\text{post}}$ , we have

$$\mathbb{E} \left[ \left( \Lambda - \hat{\lambda}_T^{\text{post}} \right)^2 \middle| \mathbf{Y} \right] = (1 - \alpha_T) \frac{1}{c} \hat{\lambda}_T^{\text{post}}.$$

- (d) Let  $\hat{\lambda}_{T-1}^{\text{post}}$  denote the posterior estimator in the sub-model where we only have observed  $(Y_1, \dots, Y_{T-1})$ . Find the credibility weight  $\beta_T \in (0, 1)$  such that the posterior estimator  $\hat{\lambda}_T^{\text{post}}$  has the recursive update structure

$$\hat{\lambda}_T^{\text{post}} = \beta_T \frac{1}{\log \frac{Y_T}{\theta}} + (1 - \beta_T) \hat{\lambda}_{T-1}^{\text{post}}.$$