Mathematical Finance

Exercise Sheet 1

Submit by 12:00 on Wednesday, September 27 via the course homepage.

Exercise 1.1 *(Path regularity and measurability)* Let $S = (S_t)_{t\geqslant 0}$ be a real-valued stochastic process. Define the processes S^* and A by $S_t^* := \sup_{0\leqslant r\leqslant t} S_r$ and $A_t := \int_0^t S_r \,\mathrm{d}r$ (when it exists), respectively.

(a) Show that if S is RCLL, then S^* is RCLL and A is well defined and continuous.

Fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions.

(b) Show that if S is RCLL and adapted, then also S^* and A are adapted.

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and define the process $\vartheta = (\vartheta_t)_{t\geqslant 0}$ by $\vartheta_t := f(S_t, S_t^*, A_t)$.

Show that if S is adapted and continuous, then ϑ is predictable.

Exercise 1.2 *(Geometric Brownian motion)* Fix constants $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion. Define the process $S = (S_t)_{t\geqslant 0}$ by

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

The process $S = (S_t)_{t\geqslant 0}$ is called a *geometric Brownian motion* and is the stock price process in the *Black–Scholes model.*

Find $\lim_{t\to\infty} S_t$ (if it exists) for all possible parameter constellations.

Hint: You may use the law of the iterated logarithm.

Exercise 1.3 *(Reparametrisation, Lemma 0.1(2))* Fix a finite time horizon $T > 0$ and let $S = (S_t)_{0\leqslant t\leqslant T}$ be a semimartingale. Prove that there is a bijection between self-financing strategies $\varphi = (\varphi^0, \vartheta)$ and pairs

$$(v_0, \vartheta) \in L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}.$$

Give explicitly the bijection map and its inverse.